Appendix

This appendix section is provided for the reviewer's information. It will not appear in the printed conference proceedings, but rather as the supplementary materias in the Proceedings Compack Disk.

As explained in [1], to model bend forces, we begin with a condition function of the form $\mathbf{C}(\mathbf{x}) = \theta$ where θ is the angle between three succesive particles on a bristle. The force this generates is $-k(\partial\mathbf{C}/\partial\mathbf{x})\mathbf{C}$, which is a sparse $3n$ vector in which only the three participating particles, i,j, and k, have non-zero entries.

Since θ is not actually a coordinate in the simulation we really have a condition of the form

$$\mathbf{C}(\mathbf{x}) = \arccos\left(\frac{(\mathbf{x}_j - \mathbf{x}_i)\cdot(\mathbf{x}_k - \mathbf{x}_j)}{||(\mathbf{x}_j - \mathbf{x}_i)||||(\mathbf{x}_k - \mathbf{x}_j)||}\right)$$

which is inconvenient for differentiating. Instead we can define an implicit function

$$\begin{aligned}\mathbf{G}(\mathbf{x}) &= \mathcal{C}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{C}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)) \\ &= \mathbf{u}\cdot\mathbf{v} - ||\mathbf{u}||||\mathbf{v}||\cos(\mathbf{C}(\mathbf{x})) = 0\end{aligned}$$

where $\mathbf{u} = \mathbf{x}_j - \mathbf{x}_i$ and $\mathbf{v} = \mathbf{x}_k - \mathbf{x}_j$. $\mathbf{G}$ and $\mathcal{C}$ are the same function, but in $\mathcal{C}$, $\mathbf{C}$ is treated as being independent of $\mathbf{x}$. From there we find $\partial\mathbf{C}/\partial\mathbf{x}_i$ by differentiating $\mathbf{G}$ with respect to $\mathbf{x}_i$ to yield:

$$\frac{\partial\mathbf{G}(\mathbf{x})}{\partial\mathbf{x}_i} = \frac{\partial\mathcal{C}}{\partial\mathbf{x}_i} + \frac{\partial\mathcal{C}}{\partial\mathbf{C}}\frac{\partial\mathbf{C}}{\partial\mathbf{x}_i} = 0$$

Rearranging,

$$\frac{\partial\mathbf{C}}{\partial\mathbf{x}_i} = -\frac{\partial\mathcal{C}/\partial\mathbf{x}_i}{\partial\mathcal{C}/\partial\mathbf{C}}$$

Performing the differentiation yields

$$\begin{aligned}\frac{\partial\mathbf{C}}{\partial\mathbf{x}_i} &= \frac{\mathbf{v} - \mathbf{u}\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}}{||\mathbf{u}\times\mathbf{v}||} \\ \frac{\partial\mathbf{C}}{\partial\mathbf{x}_k} &= \frac{\mathbf{v}\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2} - \mathbf{u}}{||\mathbf{u}\times\mathbf{v}||} \\ \frac{\partial\mathbf{C}}{\partial\mathbf{x}_j} &= -\left(\frac{\partial\mathbf{C}}{\partial\mathbf{x}_i} + \frac{\partial\mathbf{C}}{\partial\mathbf{x}_k}\right)\end{aligned}$$

Using the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

and the notation $\hat{\mathbf{x}}_q = \mathbf{x}_q/||\mathbf{x}_q||, q = i, j, k$ the first two can be rewritten as

$$\begin{aligned}\frac{\partial\mathbf{C}}{\partial\mathbf{x}_i} &= -\frac{1}{||\mathbf{u}||}\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\hat{\mathbf{v}}) \\ \frac{\partial\mathbf{C}}{\partial\mathbf{x}_k} &= -\frac{1}{||\mathbf{v}||}\hat{\mathbf{v}}\times(\hat{\mathbf{u}}\times\hat{\mathbf{v}})\end{aligned}$$

which will come as no surprise to anyone familiar with vector physics.

We wish to compute the force derivative, which can be written as

$$\frac{\partial\mathbf{f}_i}{\partial\mathbf{x}_j} = -k\left(\frac{\partial\mathbf{C}(\mathbf{x})}{\partial\mathbf{x}_i}\frac{\partial\mathbf{C}(\mathbf{x})}{\partial\mathbf{x}_j}^T + \frac{\partial^2\mathbf{C}(\mathbf{x})}{\partial\mathbf{x}_j\partial\mathbf{x}_i}\mathbf{C}(\mathbf{x})\right)$$

where we already know the first derivatives of $\mathbf{C}$. We follow the derivation of one of the nine possible second derivatives with respect to $\mathbf{x}_i$,$\mathbf{x}_j$, and $\mathbf{x}_k$, the others can be derived similarly. First we use a form of the product rule:

$$\begin{aligned}\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}_i\partial\mathbf{x}_i} &= -\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\hat{\mathbf{v}})\left(\frac{\partial}{\partial\mathbf{x}_i}\frac{1}{||\mathbf{u}||}\right)^T \\ &- \frac{1}{||\mathbf{u}||}\left(\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i}\times(\hat{\mathbf{u}}\times\hat{\mathbf{v}}) + \hat{\mathbf{u}}\times(\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i}\times\hat{\mathbf{v}})\right)\end{aligned}$$

Since each quantity of the form $\partial\hat{\mathbf{u}}/\partial\mathbf{x}_i$ is a matrix, the standard cross product does not make sense. We can, however, treat the matrix as a collection of three column vectors, and take the cross product with each in turn to form the resulting matrix. Continuing with the example, we find that

$$\frac{\partial}{\partial\mathbf{x}_i}\frac{1}{||\mathbf{u}||} \quad = \quad \frac{1}{||\mathbf{u}||}\frac{\hat{\mathbf{u}}}{||\mathbf{u}||}$$

Using that and the vector identity [] again, we rewrite the expression:

$$\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}_i\partial\mathbf{x}_i} = \frac{\partial\mathbf{C}}{\partial\mathbf{x}_i}\frac{\hat{\mathbf{u}}^T}{||\mathbf{u}||}$$
$$-\frac{1}{||\mathbf{u}||}\left(\hat{\mathbf{u}}(\hat{\mathbf{v}}^T\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i}) - \hat{\mathbf{v}}(\hat{\mathbf{u}}^T\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i}) + \frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i}(\hat{\mathbf{u}}^T\hat{\mathbf{v}}) - \hat{\mathbf{v}}(\hat{\mathbf{u}}^T\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i})\right)$$

Although a bit unorthodox, identity [] still holds if we arrange our matrices and vectors properly. With a bit of effort we find that

$$\frac{\partial\hat{\mathbf{u}}}{\partial\mathbf{x}_i} \quad = \quad -\frac{1}{||\mathbf{u}||}(\mathbf{I} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)$$

and recognize that $\hat{\mathbf{u}}$ is in the nullspce of $\partial\hat{\mathbf{u}}/\partial\mathbf{x}_i$, that is, $\hat{\mathbf{u}}^T(\partial\hat{\mathbf{u}}/\partial\mathbf{x}_i) = 0$. Thus the second derivative expression reduces to

$$\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}_i\partial\mathbf{x}_i} = \frac{\partial\mathbf{C}}{\partial\mathbf{x}_i}\frac{\hat{\mathbf{u}}^T}{||\mathbf{u}||} - \frac{1}{||\mathbf{u}||^2}\left(\hat{\mathbf{u}}(\hat{\mathbf{v}}^T(\mathbf{I} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)) + (\mathbf{I} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)(\hat{\mathbf{u}}^T\hat{\mathbf{v}})\right)$$

We can reduce this further still by multiplying out some of the inner expressions and combining like terms:

$$\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}_i\partial\mathbf{x}_i} = \frac{\partial\mathbf{C}}{\partial\mathbf{x}_i}\frac{\hat{\mathbf{u}}^T}{||\mathbf{u}||} - \frac{1}{||\mathbf{u}||^2}\left(\hat{\mathbf{u}}\hat{\mathbf{v}}^T + \hat{\mathbf{u}}^T\hat{\mathbf{v}}(\mathbf{I} - 2\hat{\mathbf{u}}\hat{\mathbf{u}}^T)\right)$$

References

[1] David Baraff and Andrew Witkin. Large steps in cloth simulation. *Proc. of ACM SIGGRAPH*, pages 43–54, 1998.