Supplementary Document: Tutorial on Nonlinear Material Model

Shan Yang and Ming C. Lin, *Fellow, IEEE*

1 BASICS

We will introduce some basic concepts in the continuum mechanics.

1.1 Stress And Stress Tensor

Stress is always simply Force/Area, but some complexity does arise because the relative orientation of the force vector to the surface normal dictates the type of stress [1]. When the force vector is normal to the surface, as shown in the Fig. 1, the stress is called normal stress and represented by σ [1]. When the force vector is parallel to the surface, the stress is called shear stress and represented by τ [1]. When the force vector is somewhere in between, then its normal and parallel components are used as follows [1].

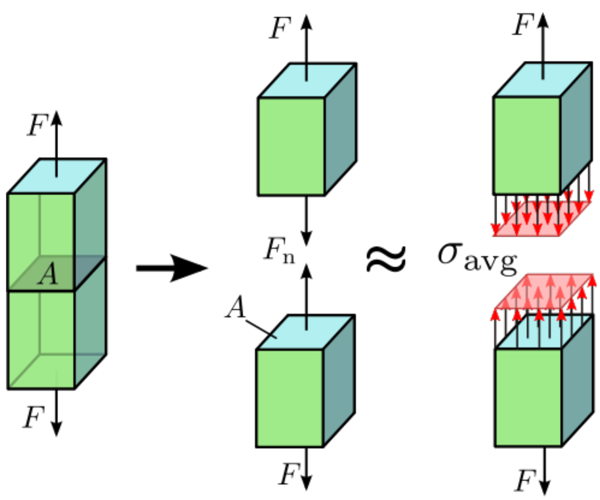

Fig. 1: The figure shows force vector normal to the surface or parallel to the surface. Copyright Wikipedia [2]

$$\sigma = \frac{F_{normal}}{A} \tag{1}$$

$$\tau = \frac{F_{parallel}}{A} \tag{2}$$

• *Shan Yang and Ming C. Lin are with the Department of Computer Science, University of North Carolina, Chapel Hill, NC, 27599-3175. E-mail: alexyang,lin@cs.unc.edu*

Manuscript received November 20, 2014.

Of course, things can get complicated in nonlinear problems with large deformations (and rotations) because the final deformed area may be different from the initial area, among other things [1].

Stress is in fact a tensor [1]. It can be written in any of several forms as follows [1]. (Cauchy stress tensor shown in Fig. 2) In 3-D it can be written as,

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{3}$$

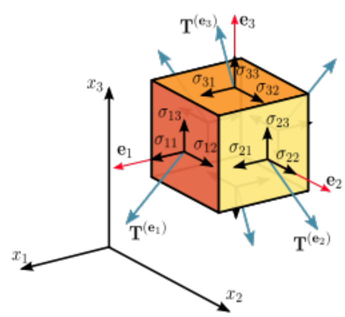

Fig. 2: The figure shows the components of the cauchy stress tensor. Copyright Wikipedia [2]

1.2 Strain And Strain Tensor

Strain like stress are been classified in to normal strains, shear strains (parallel stress). Normal in normal strain does not mean common, or usual strain [3]. It means a direct length changing stetch (or compression) of an object resulting from a normal stress [3]. It is defined as (the quantities are shown in Fig. 3),

$$\epsilon = \frac{\Delta L}{L_o} \tag{4}$$

This is also known as Engineering Strain [3].

Shear strain is usually represented by γ and defined as [3],

$$\gamma = \frac{D}{T} \tag{5}$$

This is the shear-version of engineering strain [3].

Strain, like stress, is a tensor [3]. And like stress, strain is a tensor simply because it obeys the standard coordinate

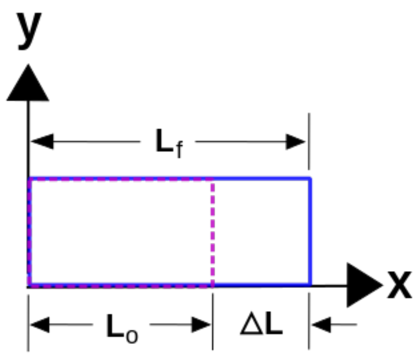

Fig. 3: The figure shows the stetch from a normal stress and the resulted normal strain. Copyright Continuum Mechanics [3]

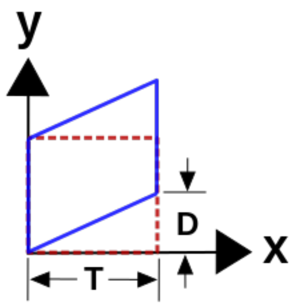

Fig. 4: The figure shows the shear strain definition. Copyright Continuum Mechanics [3]

transformation principles of tensors [3]. It can be written in any of several different forms as follows [3].

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_{zz} \end{bmatrix} \tag{6}$$

1.3 Deformation Gradient

Displacement and deformations are the essentials of continuum mechanics. The deformation gradient is used to separate rigid body translations and rotations from deformations, which are the source of stresses [4]. In this tutorial we won't be covering rigid body dynamics. As is the convention in continuum mechanics, the vector $\mathbf{X}$ is used to define the undeformed reference configuration, and $\mathbf{x}$ defines the deformed current configuration [4]. The deformation gradient $\mathbf{F}$ (shown in Fig. 5) is the derivative of each component of the deformed $\mathbf{x}$ vector with respect to each component of the reference $\mathbf{X}$ vector [4], then

$$F_{ij} = x_{i,j} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{7}$$

The displacement $\mathbf{u}$ can be defined as,

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{8}$$

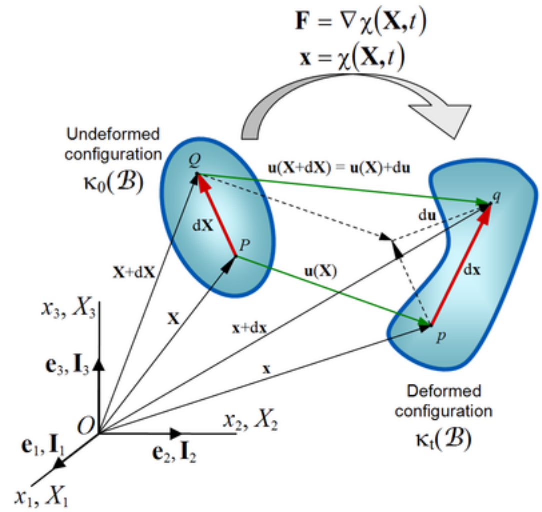

Fig. 5: The figure shows the displacement field and the deformation gradient. Copyright Wikipedia [5]

2 Nonlinear Material Model

For many materials, linear elastic models cannot accurately capture the observed material behavior; hyperelastic material models that can capture the nonlinear material behavior subjected to large strain. For example, animal tissue and some common organic materials are commonly been simulated using hyperelastic material.

Material model describes the behavior of a deformable body by defining the relation between the displacement field and the stress. Through the material model, we can compute the displacement field given the stress or vice versa. In order to define a nonlinear material model, we need to define the following,

1) Displacement-Strain Relation
2) Energy-Strain Relation
3) Strain-Stress Relation

2.1 Displacement-Strain Model

The displacement-strain model describes the relation between the displacement and the strain. In this section we will introduce the Green-Lagrange strain model. Before we introduce the strain model, we first set some notations. We will use small $\mathbf{n}$ as the surface normal for the deformed configuration while the $\mathbf{N}$ for that of the reference configuration. The Green-Lagrange strain model is designed for the measurement of large strain. It is defined through the right Cauchy strain tensor $\mathbf{C}_r = \mathbf{F}^T\mathbf{F}$, where $\mathbf{F}$ is the deformation gradient. The right Cauchy strain tensor measures the square of the changes of local deformation. The Green-Lagrange strain tensor $\mathbf{E}$ removes the rigid body transformation from the right Cauchy strain tensor. It is defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}_r - \mathbf{I}) \tag{9}$$

Specifically each element of the strain tensor matrix $\mathbf{E}$

$$\mathbf{E}_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \mathbf{X}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{X}_i} + \frac{\partial \mathbf{u}_k}{\partial \mathbf{X}_i}\frac{\partial \mathbf{u}_k}{\partial \mathbf{X}_j}\right) \tag{10}$$

It basically consists two parts, the small strain terms and the quadratic terms as shown in Eqn. 10. When the strain is small the quadratic terms can be ignored, and the Green-Lagrange strain behaves the same as the Engineering strain model which only contains the first part. But when the strain is large the quadratic terms record the strain. The quadratic terms also accounts for the **geometric non-linearity** of the strain-displacement relations.

2.2 Energy-Strain Model

The internal energy of an object consists of thermal energy and elastic strain energy. For hyperelastic material model, the variation of the thermal energy is neglected. The stress-strain relation for a nonlinear material model is defined through the strain energy. The strain energy is the work done by the stress as is shown in Fig. 6.

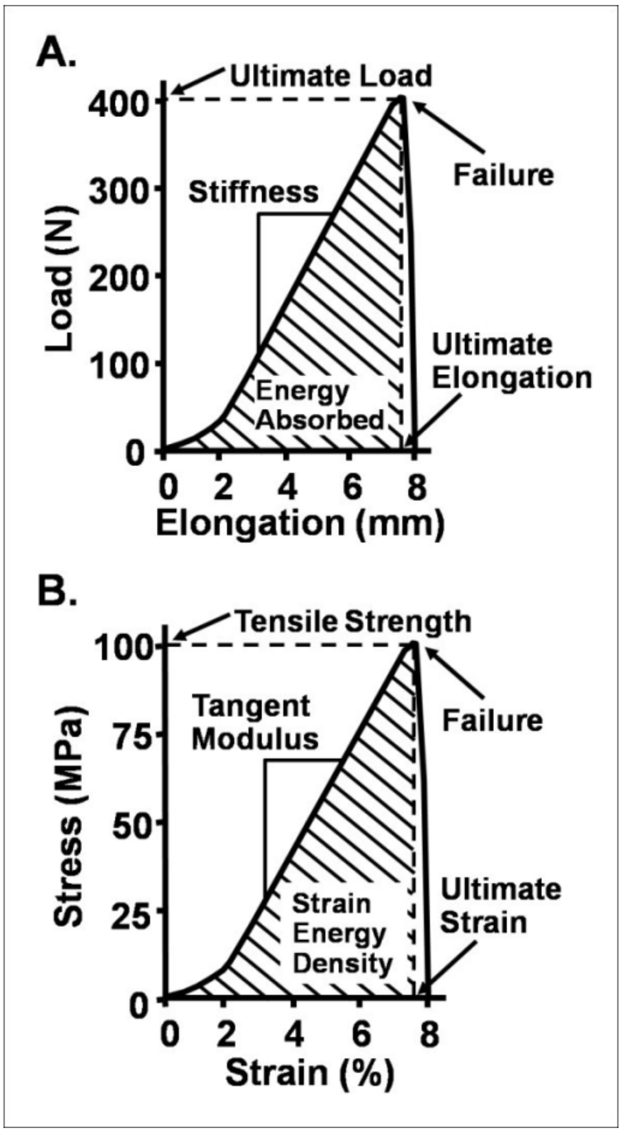

Fig. 6: The figure shows force vector normal to the surface or parallel to the surface. Copyright BioMed Central Ltd [6]

The energy density function determines the behavior of the deformable object when subjected to stress. The energy density function is essentially a mapping from the stretches to the energy. For a material model to be isotropic in general, the energy function is expressed as a function of the invariants $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ of strain tensor. The invariants are computed from the principal stretches. When we do polar decomposition on the deformation gradient $\mathbf{F}$, we obtain,

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{11}$$

in which the matrix $\mathbf{R}$ is orthogonal. It is the rotation matrix. The matrix $\mathbf{U}$ and the matrix $\mathbf{V}$ have the same eigenvalues. Those eigenvalues are the principal stretches λ.

$$\begin{aligned} \mathbf{I}_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ \mathbf{I}_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \\ \mathbf{I}_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{12}$$

If the material is incompressible, the third invariant $\mathbf{I}_3$ equals to one.

One general energy function of this class of incompressible material proposed by Rivlin [7] is,

$$\mathbf{\Psi_R} = \sum_{i,j=0}^{\infty} \mathbf{C}_{ij}(\mathbf{I}_1 - 3)^i(\mathbf{I}_2 - 3)^j \tag{13}$$

where $\mathbf{C}_{ij}$ are the material parameters. Many classical material model is derived from this. One obtains the neo-Hookean model by keeping the first term of the Rivlin expression.

$$\mathbf{\Psi_{NH}} = \mathbf{C}_{10}(\mathbf{I}_1 - 3) \tag{14}$$

The classic Mooney-Rivlin model [8] is,

$$\mathbf{\Psi_{MR}} = \mathbf{C}_{10}(\mathbf{I}_1 - 3) + \mathbf{C}_{01}(\mathbf{I}_2 - 3) \tag{15}$$

By adding the second term, the Mooney-Rivlin model can better describe the uniaxial tension behavior. To better capture the behavior of larger stretches, researchers use higher order of $\mathbf{I}_1$. One such model is the Yeoh model [9],

$$\mathbf{\Psi_Y} = \mathbf{C}_{10}(\mathbf{I}_1 - 3) + \mathbf{C}_{20}(\mathbf{I}_1 - 3)^2 + \mathbf{C}_{30}(\mathbf{I}_1 - 3)^3 \tag{16}$$

To account for volume changes, compressible forms of this class of material are proposed by adding the third principle to the Rivlin expression Eqn. 13.

$$\mathbf{\Psi} = \mathbf{\Psi_R} + \mathbf{\Psi}(\mathbf{J}) \tag{17}$$

where $\mathbf{J}$ is the volume ratio $\mathbf{J} = \sqrt{\mathbf{I}_3}$. In this paper, we use this form of energy function of Mooney-Rivlin material [10], [11]:

$$\mathbf{\Psi} = \frac{1}{2}w_1((\mathbf{I}_1^2 - \mathbf{I}_2)/\mathbf{I}_3^{\frac{2}{3}} - 6) + w_2(\mathbf{I}_1/\mathbf{I}_3^{\frac{1}{3}} - 3) + v_1(\mathbf{I}_3^{\frac{1}{2}} - 1)^2. \tag{18}$$

where w_1, w_2 and v_1 are the material parameters. The first two elasticity parameters, w_1 and w_2, are related to distortional response (i.e., together they describe the response of the material when subject to shear stress, uniaxial stress and equibiaxial stress), while the last parameter, v_1, is related to volumetric response (i.e. it describes the material response to bulk stress). $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are the three invariants. The invariants are computed from the principal stretches, which are the corresponding singular values of the deformation gradient $\mathbf{F}$.

2.3 Strain-Stress Model

We will be using the second Piola-Kirchhoff stress tensor with the Green-Lagrange strain tensor. The second Piola-Kirchhoff stress $\boldsymbol{\sigma}^{\text{PK2}}$ tensor for hyperelastic material is defined through the energy function and the Green-Lagrange strain tensor $\mathbf{E}$. Thus the Green-Lagrange strain and the second Piola-Kirchhoff stress relation is defined as,

$$\boldsymbol{\sigma}^{\text{PK2}} = \frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{E}} \tag{19}$$

Again the energy density $\boldsymbol{\Psi}$ for hyperelastic material contains only the elastic energy. We can now give the deformation energy computed from the Green-Lagrange strain tensor $\mathbf{E}$ and the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{\text{PK2}}$ of the deformable body Ω,

$$\int_{\Omega} \boldsymbol{\sigma}^{\text{PK2}} : \mathbf{E} \, \mathrm{d}\Omega \tag{20}$$

References

[1] B. McGinty, www.continuummechanics.org/cm/stress.html, 2012.
[2] J. Stolfi, https://en.wikipedia.org/wiki/Stress_(mechanics), 2013.
[3] B. McGinty, www.continuummechanics.org/cm/strain.html, 2012.
[4] ——, www.continuummechanics.org/cm/deformationgradient.html, 2012.
[5] Sanpaz, https://en.wikipedia.org/wiki/Finite_strain_theory, 2008.
[6] H.-J. Jung, M. B. Fisher, and S. L. Woo, "Role of biomechanics in the understanding of normal, injured, and healing ligaments and tendons," *BMC Sports Science, Medicine and Rehabilitation*, vol. 1, no. 1, p. 9, 2009.
[7] R. Rivlin, "Large elastic deformations of isotropic materials. iv. further developments of the general theory," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 241, no. 835, pp. 379–397, 1948.
[8] M. Mooney, "A theory of large elastic deformation," *Journal of applied physics*, vol. 11, no. 9, pp. 582–592, 1940.
[9] O. Yeoh, "Some forms of the strain energy function for rubber," *Rubber Chemistry and technology*, vol. 66, no. 5, pp. 754–771, 1993.
[10] L. R. Treloar, H. Hopkins, R. Rivlin, and J. Ball, "The mechanics of rubber elasticity [and discussions]," *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. 351, no. 1666, pp. 301–330, 1976.
[11] R. S. Rivlin and D. Saunders, "Large elastic deformations of isotropic materials. vii. experiments on the deformation of rubber," *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 243, no. 865, pp. 251–288, 1951.