

Curves

- Locus of a point moving with one degree of freedom
- Locus of a one-dimensional parameter family of point
- **Mathematically** defined using:
 - Explicit equations
 - Implicit equations
 - Parametric equations (Hermite, Bezier, B-spline)



Geometric Modeling of Curves

Computational Representations of a Curve for:

- Data fitting applications
- Shape representation (e.g. font design)
- Intersection computations



Explicit Equations

$$Y = f(x)$$

- There is only one y value for each x value; not vice-versa
- Easy to generate points or plot of the curves
- Can easily check whether a point lies on the curve
- Cannot represent closed or multiple-valued curves



Implicit Equations

Can represent closed form or multiple-valued:
 $f(x,y)=0$

- Mostly deal with polynomial or rational functions
- Implicits are a proper superset of rational parametric

E.g. **Line**: $Ax + By + C = 0$

Conic: $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$

The coefficients determine the geometric properties



Parametric Equations of Curves

$$P(u) = [x(u) \quad y(u) \quad z(u)]$$

where $x()$, $y()$ and $z()$ are polynomial or rational functions. The definition extends to N-dimensions

- Usually the domain is restricted to $u \in [0,1]$ or a subset of real domain
- Each piece is a curve segment

$Q(u,v) = [x(u,v) \quad y(u,v) \quad z(u,v)]$ is a surface

$P()$ and $Q()$ are vector valued functions

Partials of $P()$ & $Q()$ are used to compute tangents and normals to the curves & surfaces



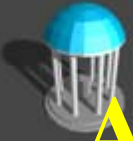
Parametric Equations of Curves

- **Model Space:** x, y, z Cartesian
- **Parametric space:** u, v space or parametric domain
- **Direct Mapping:** Parametric \Rightarrow Model space
 - Involves function evaluation
- **Inverse mapping or Inversion:** Given (x, y, z) compute u or (u, v)
 - Involves solving non-linear equations
- **Reparametrization:** To change the parametric domain or interval used to define the curve



Advantages of Parametric Formulation

- Allow separation of variables & direction computation of point coordinates
- Easy to express them as vectors
- Each variable is treated alike
- More degrees of freedom to control curve shape
- Transformations can be performed directly on the curves
- Accommodate slopes without computational breakdown



Advantages of Parametric Formulation

- Extension or contraction to higher or lower dimension is direct
- The curves are inherently bounded when the parameter is constrained to a specified finite interval
- Same curve can be represented by multiple parametrizations
- Choice of parametrization, because of computational properties or application related benefits



Conic Curves

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

has a matrix form

$$\mathbf{P} \mathbf{Q} \mathbf{P}^T = 0,$$

where

$$\mathbf{P} = [x \ y \ 1], \ \&$$

$$\mathbf{Q} = \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}$$

\mathbf{P} is given by homogeneous coordinates



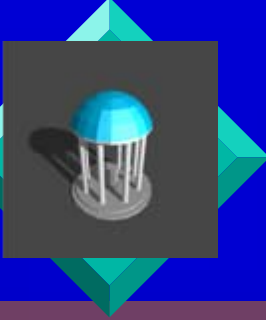
Conic Curves

- Many characteristics are invariant under translation and rotation transformation
- These include, $A + C$, $k = AC - B^2$, and the determinant of Q
- The values of k and Q indicate the type of conic curve
- Common conics are parabola, hyperbola and ellipse



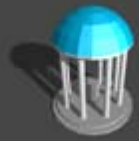
Parametric Curves

- **Hermite Curves:** based on Hermite interpolation; uses points & derivative data
- **Bezier Curves:** Defined by control points which determine its degree; interpolates the first & last point; no local control
- **B-Spline Curves:** piecewise polynomial or rational curve defined by control points; need not interpolate any point; degree is independent of the number of control points; local control; affine invariance



Composing Parametric Curves

- Given a large collection of data points, compute a curve representation that approximates or interpolates
- Higher degree curves (say more than 4 or 5) can result in numerical problems (evaluation, intersection, subdivision etc.)
- Need to multiple segments and compose them with appropriate continuity



Parametric & Geometric Continuity

- Parametric Continuity (or C^n): Two curves have n th order parametric continuity, C^n if their 0^{th} to n^{th} derivatives match at the end points
- Geometric Continuity (or G^n): Less restrictive than parametric continuity. Two curves have n th order geometric continuity, G^n if there is a reparametrization of the curve, so that the reparametrized curves have C^n continuity.
 - G^1 : Unit tangent vectors at the end point are continuous
 - G^2 : Relates the curvature of the curves at the endpoints
 - Geometric continuity results in more degrees of freedom