Curves

- Locus of a point moving with one degree of freedom
- Locus of a one-dimensional parameter family of point
- Mathematically defined using:
  - Explicit equations
  - Implicit equations
  - Parametric equations (Hermite, Bezier, B-spline)

Geometric Modeling of Curves

Computational Representations of a Curve for:

- Data fitting applications
- Shape representation (e.g. font design)
- Intersection computations

Explicit Equations

$Y = f(x)$

- There is only one y value for each x value; not vice-versa
- Easy to generate points or plot of the curves
- Can easily check whether a point lies on the curve
- Cannot represent closed or multiple-valued curves

Implicit Equations

Can represent closed form or multiple-valued: $f(x,y)=0$

- Mostly deal with polynomial or rational functions
- Implicits are a proper superset of rational parametric

E.g. Line: $Ax + By + C = 0$

Conic: $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$

The coefficients determine the geometric properties

Parametric Equations of Curves

P(u) = [x(u)   y(u)   z(u)]

where x( ), y( ) and z( ) are polynomial or rational functions. The definition extends to N-dimensions

- Usually the domain is restricted to u $\in$ [0,1] or a subset of real domain
- Each piece is a curve segment

Q(u,v) = [x(u,v)   y(u,v)   z(u,v)] is a surface

P( ) and Q( ) are vector valued functions

Partials of P( ) & Q( ) are used to compute tangents and normals to the curves & surfaces

Parametric Equations of Curves

- Model Space: x,y,z Cartesian
- Parametric space: u,v space or parametric domain

- Direct Mapping: Parametric => Model space
  - Involves function evaluation
- Inverse mapping or Inversion: Given (x,y,z) compute u or (u,v)
  - Involves solving non-linear equations
- Reparametrization: To change the parametric domain or interval used to define the curve

Advantages of Parametric Formulation

- Allow separation of variables & direction computation of point coordinates
- Easy to express them as vectors
- Each variable is treated alike
- More degrees of freedom to control curve shape
- Transformations can be performed directly on the curves
- Accommodate slopes without computational breakdown

Advantages of Parametric Formulation

- Extension or contraction to higher or lower dimension is direct
- The curves are inherently bounded when the parameter is constrained to a specified finite interval
- Same curve can be represented by mulitiple parametrizations
- Choice of parametrization, because of computational properties or application related benefits

Conic Curves

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

has a matrix form

$$P\,Q\,P^T = 0,$$

where

$$P = [\,x\ y\ 1\,],\ \&$$

$$Q = \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}$$

P is given by homogeneous coordinates

Conic Curves

- Many characteristics are invariant under translation and rotation transformation
- These include, A + C, $k = AC - B^2$, and the determinant of Q
- The values of k and Q indicate the type of conic curve
- Common conics are parabola, hyperbola and ellipse

Parametric Curves

- Hermite Curves: based on Hermite interpolation; uses points & derivative data
- Bezier Curves: Defined by control points which determine its degree; interpolates the first & last point; no local control
- B-Spline Curves: piecewise polynomial or rational curve defined by control points; need not interpolate any point; degree is independent of the number of control points; local control; affine invariance

Composing Parametric Curves

- Given a large collection of data points, compute a curve representation that approximates or interpolates
- Higher degree curves (say more than 4 or 5) can result in numerical problems (evaluation, intersection, subdivision etc.)
- Need to multiple segments and compose them with appropriate continuity

Parametric & Geometric Continuity

- Parametric Continuity (or C^n): Two curves have nth order parametric continuity, C^n, if their 0^{th} to n^{th} derivatives match at the end points
- Geometric Continuity (or G^n): Less restrictive than parametric continuity. Two curves have nth order geometric continuity, G^n, if there is a reparametrization of the curve, so that the reparametrized curves have C^n continuity.
  - G^1: Unit tangent vectors at the end point are continuous
  - G^2: Relates the curvature of the curves at the endpoints
  - Geometric continuity results in more degrees of freedom