COMP 790-058

(Based on Slides from J. Latombe @ Stanford)

Configuration Space
What is a Path?
Tool: Configuration Space (C-Space C)
Configuration Space

\[ q = (q_1, \ldots, q_n) \]
Definition

A robot configuration is a specification of the positions of all robot points relative to a fixed coordinate system.

Usually a configuration is expressed as a “vector” of position/orientation parameters.
• 3-parameter representation: \( q = (x, y, \theta) \)
• In a 3-D workspace \( q \) would be of the form \( (x, y, z, \alpha, \beta, \gamma) \)
Articulated Robot Example

\[ q = (q_1, q_2, \ldots, q_{10}) \]
Protein example
Configuration Space of a Robot

- Space of all its possible configurations
- But the topology of this space is usually not that of a Cartesian space
Configuration Space of a Robot

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\[ C = S^1 \times S^1 \]
Configuration Space of a Robot

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\[ C = S^1 \times S^1 \]
What is its Topology?

\((S1)^7 \times \mathbb{R}^3\)
Structure of Configuration Space

- It is a manifold
  For each point $q$, there is a 1-to-1 map between a neighborhood of $q$ and a Cartesian space $\mathbb{R}^n$, where $n$ is the dimension of $C$

- This map is a local coordinate system called a chart.
  $C$ can always be covered by a finite number of charts. Such a set is called an atlas
Example
Case of a Planar Rigid Robot

- 3-parameter representation: \( q = (x, y, \theta) \) with \( \theta \in [0, 2\pi) \). Two charts are needed.
- Other representation: \( q = (x, y, \cos \theta, \sin \theta) \)
  \( \rightarrow \) c-space is a 3-D cylinder \( \mathbb{R}^2 \times S^1 \) embedded in a 4-D space.
Rigid Robot in 3-D Workspace

- \( q = (x, y, z, \alpha, \beta, \gamma) \)

The c-space is a 6-D space (manifold) embedded in a 12-D Cartesian space. It is denoted by \( \mathbb{R}^3 \times \text{SO}(3) \)

where \( r_{11}, r_{12}, \ldots, r_{33} \) are the elements of rotation matrix \( R \):

\[
\begin{pmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{pmatrix}
\]

with:

- \( r_{i1}^2 + r_{i2}^2 + r_{i3}^2 = 1 \)
- \( r_{i1}r_{j1} + r_{i2}r_{2j} + r_{i3}r_{j3} = 0 \)
- \( \det(R) = +1 \)
Parameterization of $\text{SO}(3)$

- **Euler angles**: $(\phi, \theta, \psi)$
- **Unit quaternion**: $(\cos \theta/2, n_1 \sin \theta/2, n_2 \sin \theta/2, n_3 \sin \theta/2)$
Metric in Configuration Space

A metric or distance function $d$ in $C$ is a map
$$d: \ (q_1,q_2) \in C^2 \rightarrow d(q_1,q_2) \geq 0$$
such that:
- $d(q_1,q_2) = 0$ if and only if $q_1 = q_2$
- $d(q_1,q_2) = d(q_2,q_1)$
- $d(q_1,q_2) \leq d(q_1,q_3) + d(q_3,q_2)$
Metric in Configuration Space

Example:

- Robot $A$ and point $x$ of $A$
- $x(q)$: location of $x$ in the workspace when $A$ is at configuration $q$
- A distance $d$ in $C$ is defined by:
  \[ d(q, q') = \max_{x \in A} ||x(q) - x(q')|| \]
  where $||a - b||$ denotes the Euclidean distance between points $a$ and $b$ in the workspace
Specific Examples in $\mathbb{R}^2 \times S^1$

- $q = (x,y,\theta), \; q' = (x',y',\theta')$ with $\theta, \theta' \in [0,2\pi)$
- $\alpha = \min\{|\theta-\theta'|, \; 2\pi-|\theta-\theta'|\}$

\[ d(q,q') = \sqrt{(x-x')^2 + (y-y')^2 + \alpha^2} \]

\[ d(q,q') = \sqrt{(x-x')^2 + (y-y')^2 + (\alpha \rho)^2} \]

where $\rho$ is the maximal distance between the reference point and a robot point
Notion of a Path

- A path in $C$ is a piece of continuous curve connecting two configurations $q$ and $q'$:
  \[ \tau : s \in [0,1] \rightarrow \tau(s) \in C \]

- $s' \rightarrow s \Rightarrow \text{d}(\tau(s),\tau(s')) \rightarrow 0$
Other Possible Constraints on Path

- Finite length, smoothness, curvature, etc...
- A trajectory is a path parameterized by time:
  \[ \tau : t \in [0,T] \rightarrow \tau(t) \in C \]
Obstacles in C-Space

- A configuration $q$ is **collision-free**, or **free**, if the robot placed at $q$ has null intersection with the obstacles in the workspace.

- The **free space** $F$ is the set of free configurations.

- A **C-obstacle** is the set of configurations where the robot collides with a given workspace obstacle.

- A configuration is **semi-free** if the robot at this configuration touches obstacles without overlap.
Disc Robot in 2-D Workspace
Rigid Robot Translating in 2-D

$CB = B \ominus A = \{b-a \mid a \in A, b \in B\}$
Rigid Robot Translating in 2-D

\[ CB = B \Theta A = \{ b-a \mid a \in A, b \in B \} \]
Linear-Time Computation of C-Obstacle in 2-D (convex polygons)
Rigid Robot Translating and Rotating in 2-D
C-Obstacle for Articulated Robot
Free and Semi-Free Paths

- A **free path** lies entirely in the free space $F$
- A **semi-free path** lies entirely in the semi-free space
Remark on Free-Space Topology

- The robot and the obstacles are modeled as closed subsets, meaning that they contain their boundaries.
- One can show that the C-obstacles are closed subsets of the configuration space C as well.
- Consequently, the free space F is an open subset of C. Hence, each free configuration is the center of a ball of non-zero radius entirely contained in F.
- The semi-free space is a closed subset of C. Its boundary is a superset of the boundary of F.
Notion of Homotopic Paths

- Two paths with the same endpoints are **homotopic** if one can be continuously deformed into the other.

- $R \times S^1$ example:

- $\tau_1$ and $\tau_2$ are homotopic.
- $\tau_1$ and $\tau_3$ are not homotopic.
- In this example, infinity of homotopy classes
Connectedness of $C$-Space

- $C$ is **connected** if every two configurations can be connected by a path.

- $C$ is **simply-connected** if any two paths connecting the same endpoints are homotopic.  
  Examples: $\mathbb{R}^2$ or $\mathbb{R}^3$

- Otherwise $C$ is **multiply-connected**  
  Examples: $S^1$ and $SO(3)$ are multiply-connected:  
  - In $S^1$, infinity of homotopy classes  
  - In $SO(3)$, only two homotopy classes