Supplementary Material for Paper: Free-Flowing Granular Materials with Two-Way Solid Coupling

1 Staggered projections

The pressure p and frictional stress $\mathbf{s}$ are defined by a pair of coupled quadratic programs,

$$\min F(p) : p \geq 0, \tag{1}$$

$$\min E(\mathbf{s}) : -s_{\max} \leq s_{ij} \leq s_{\max}. \tag{2}$$

Below, we show that applying staggered projections has the simultaneous solution of both these minimizations as a fixed point. This proof closely follows Section 6 of [Kaufman et al. 2008], but works in the continuous setting.

We reinterpret these above minimizations as projections in impulse space. To simplify the discussion, let us introduce symbols to denote the predicted momentum of the material, and the impulses imparted by pressure and friction:

$$\boldsymbol{\mu}_0 = \rho^n \left(\mathbf{v}^n + \frac{\Delta t}{\rho^n} \mathbf{f}_{\text{ext}} \right), \tag{3}$$

$$\boldsymbol{\mu}_p = -\Delta t \nabla p, \tag{4}$$

$$\boldsymbol{\mu}_s = \Delta t \nabla \cdot \mathbf{s}. \tag{5}$$

We define the projection of some impulse $\boldsymbol{\mu}$ to its nearest point in a convex set A as

$$P(\boldsymbol{\mu}; A) = \operatorname*{argmin}_{\boldsymbol{\nu} \in A} \int \|\boldsymbol{\nu} - \boldsymbol{\mu}\|^2 \mathrm{d}V \tag{6}$$

$$= \operatorname*{argmin}_{\boldsymbol{\nu} \in A} \int \left(\frac{1}{2} \|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}^T \boldsymbol{\mu} \right) \mathrm{d}V. \tag{7}$$

This projection operation is a non-expansive mapping under the Euclidean metric.

If we consider the linear term b_1 of the pressure solve to be proportional to the divergence of a vector field $\boldsymbol{\beta}$, the objective functional $F(p)$ can be readily shown to be equivalent to

$$F(p) = \frac{\Delta t^2}{\rho_{\max}} \int \left(\frac{1}{2} \|\nabla p\|^2 + b_1 p \right) \mathrm{d}V \tag{8}$$

$$= \frac{1}{\rho_{\max}} \int \left(\frac{1}{2} \|\boldsymbol{\mu}_p\|^2 - \boldsymbol{\beta}^T \boldsymbol{\mu}_p \right) \mathrm{d}V, \tag{9}$$

where $\boldsymbol{\beta}$ satisfies $\frac{\Delta t}{\rho_{\max}} \nabla \cdot \boldsymbol{\beta} = b_1 = (1 - \phi^{n+1}|_{p=0})$. But as

$$\phi^{n+1}|_{p=0} = \left(\phi^n + \frac{\Delta t}{\rho_{\max}} \nabla \cdot \boldsymbol{\mu}_0 \right) + \frac{\Delta t}{\rho_{\max}} \nabla \cdot \boldsymbol{\mu}_s, \tag{10}$$

this implies that $\boldsymbol{\beta}$ is simply a constant, say $\boldsymbol{\beta}_0$, minus $\boldsymbol{\mu}_s$. Therefore, minimizing $F(p)$ is equivalent to minimizing $F'(p) = \int \|\boldsymbol{\mu}_p - \boldsymbol{\beta}_0 + \boldsymbol{\mu}_s\|^2 \mathrm{d}V$, and the pressure solve can be expressed as a projection

$$\boldsymbol{\mu}_p = P(\boldsymbol{\beta}_0 - \boldsymbol{\mu}_s \, ; \, A_1) \tag{11}$$

onto the convex set $A_1 = \{\nabla p : p \geq 0\}$.

The friction solve is more straightforward. Its objective function is simply

$$E(\mathbf{s}) = \frac{1}{\rho_{\max}} \int \left(\frac{1}{2} \|\boldsymbol{\mu}_s\|^2 + \boldsymbol{\mu}_s^T (\boldsymbol{\mu}_0 + \boldsymbol{\mu}_p) \right) \mathrm{d}V, \tag{12}$$

so the frictional impulse is the projection

$$\boldsymbol{\mu}_s = P(-\boldsymbol{\mu}_0 - \boldsymbol{\mu}_p \, ; \, A_2) \tag{13}$$

onto the set $A_2 = \{\nabla \cdot \mathbf{s} : -s_{\max} \leq s_{ij} \leq s_{\max}\}$.

By substituting (13) into (11), we obtain the fixed-point property

$$\boldsymbol{\mu}_p = P(\boldsymbol{\beta}_0 - P(-\boldsymbol{\mu}_0 - \boldsymbol{\mu}_p \, ; \, A_2) \, ; \, A_1) \tag{14}$$

which characterizes the solutions to the stress response. In fact, since the right-hand side of the above is a composition of projections, it is a non-expansive mapping and is often contractive. Thus, iteratively applying the projections (13) and (11) in a staggered sequence, or equivalently, solving each quadratic program (1) and (2) in turn, is a valid method for solving the coupled system.

2 Separation of components

Solving the friction projection (13) involved a matrix D_2 representing the tensor gradient $\nabla \cdot \mathbf{s}$, which for a trace-free tensor field $\mathbf{s}$ is 5 times larger than the corresponding to the gradient of a scalar p. Directly solving the quadratic program using this full system leads to numerical difficulties and poor convergence. Instead, we perform the minimization on each component of $\mathbf{s}$ in turn.

That is, first we minimize E with respect to the component s_{yy}, holding all other components fixed. The minimization is then of the form

$$E = E|_{s_{yy}=0} + \frac{1}{2} \mathsf{s}_{yy}^T \mathsf{A}_{yy} \mathsf{s}_{yy} + \mathsf{b}_{yy}^T \mathsf{s}_{yy}, \tag{15}$$

$$\mathsf{A}_{yy} = \frac{\Delta t^2}{\rho_{\max}} \mathsf{D}_y^T \mathsf{D}_y, \tag{16}$$

$$\mathsf{b}_{yy} = \frac{\Delta t}{\rho_{\max}} \mathsf{D}_y^T \rho^n \tilde{\mathbf{v}}|_{s_{yy}=0}. \tag{17}$$

This simply involves the finite difference matrix D_y corresponding to ∂/∂_y, which is numerically much more well-behaved. (Note that in the absence of solid bodies, the components of s_{yy} at different x and z positions are decoupled, and each column can be solved independently. This does not hold when inteacting solid bodies are present.) After this, s_{xx} and s_{zz} are solved together, using the trace-free condition $s_{xx} + s_{zz} = -s_{yy}$. Similarly the diagonal components s_{xy}, s_{yz} and s_{xz} are determined in turn, holding previously solved components fixed at their updated values.

This amounts to performing minimization over a set of orthogonal subspaces that span the space of frictional stresses. Since each minimization is a projection, and the coupled solution is a fixed point of each of them, a staggered projection sequence that solves for p, s_{yy}, s_{xx} & s_{zz}, s_{xy}, s_{yz}, and s_{xz} in turn remains a non-expansive mapping with the coupled solution to (1) and (2) as its fixed point. This is the solution procedure that we use in our implementation.

References

KAUFMAN, D. M., SUEDA, S., JAMES, D. L., AND PAI, D. K. 2008. Staggered projections for frictional contact in multibody systems. *ACM Transactions on Graphics (SIGGRAPH Asia 2008) 27*, 5, 164:1–164:11.