Sampling and reconstruction

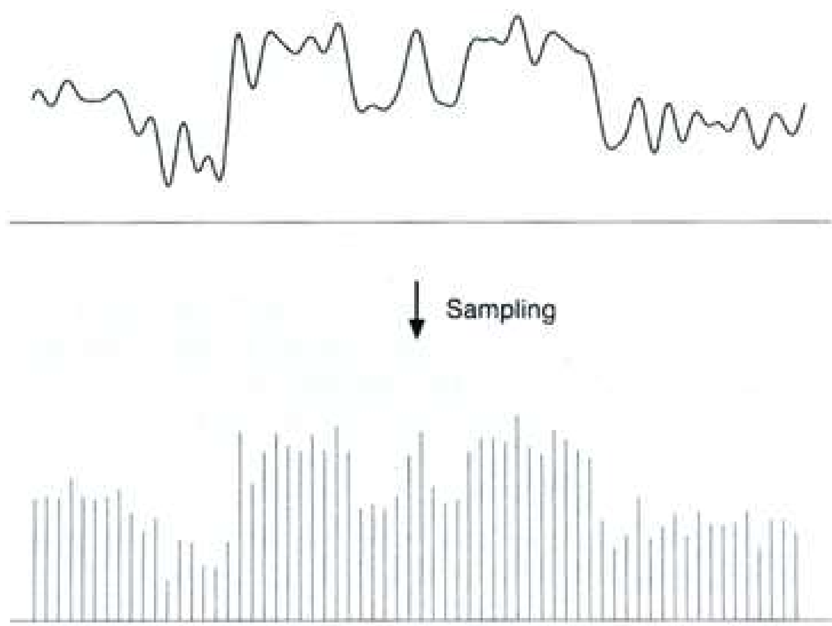
COMP 575/COMP 770

Spring 2016

Sampled representations

- How to store and compute with continuous functions?
- Common scheme for representation: samples

write down the function’s values at many points



[FvDFH fig.14.14b / Wolberg]

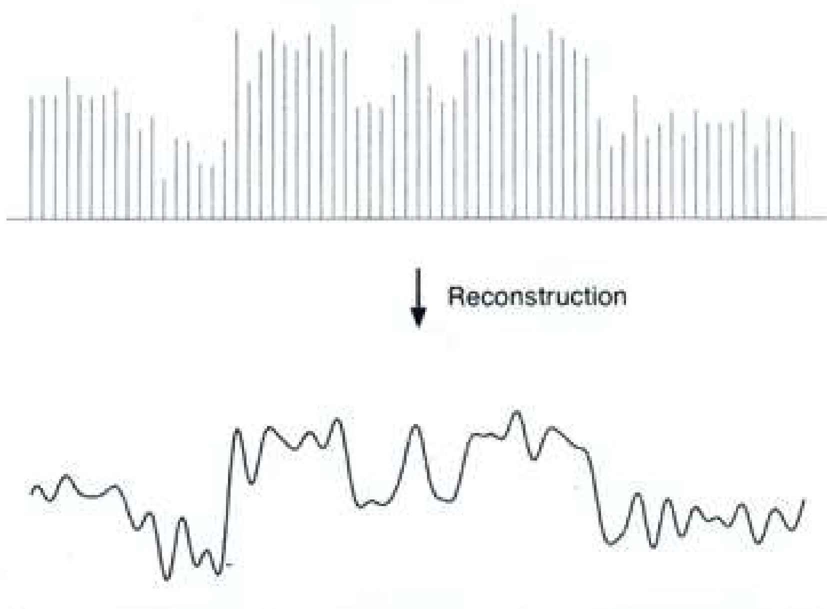
Reconstruction

- Making samples back into a continuous function

for output (need realizable method)

for analysis or processing (need mathematical method)

amounts to “guessing” what the function did in between



[FvDFH fig.14.14b / Wolberg]

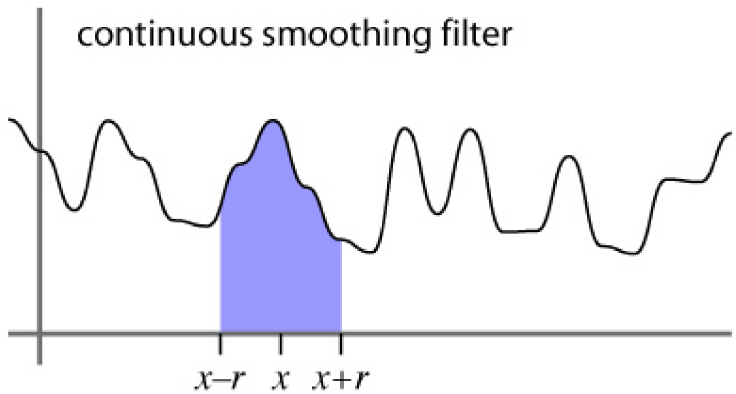
Filtering

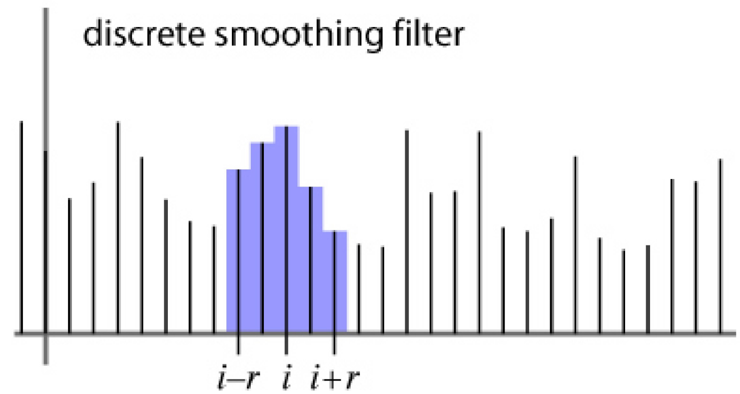
- Processing done on a function

can be executed in continuous form (e.g. analog circuit)

but can also be executed using sampled representation

- Simple example: smoothing by averaging





Roots of sampling

- Nyquist 1928; Shannon 1949

famous results in information theory

- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc

the first high-profile consumer application

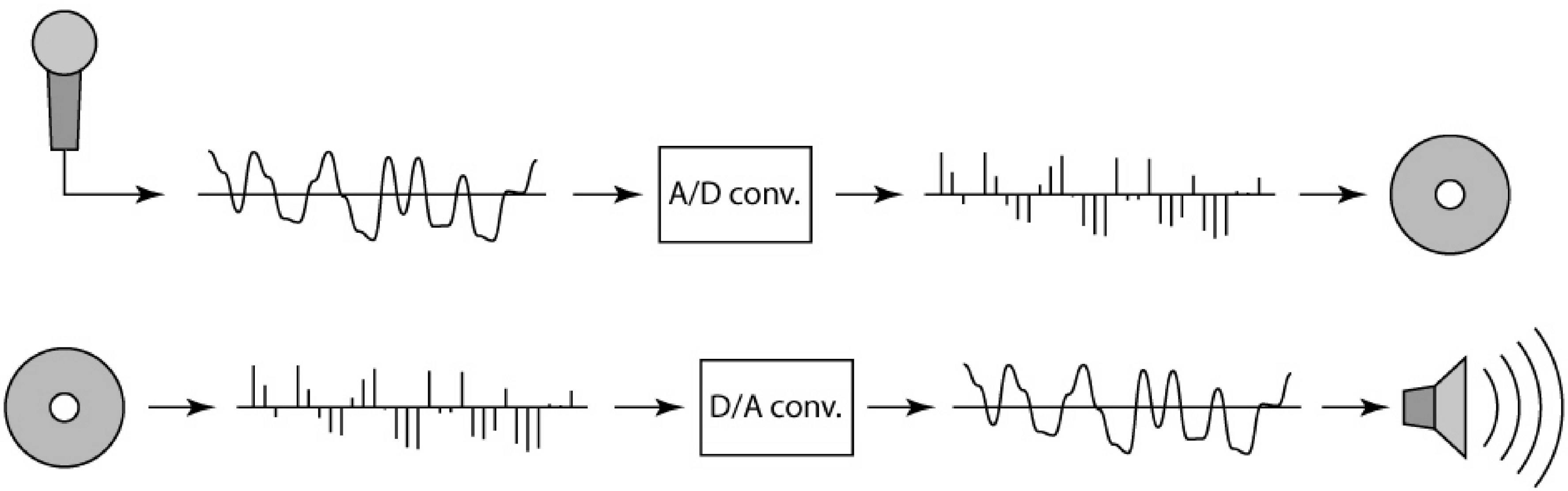
- This is why all the terminology has a communications or audio "flavor"

early applications are 1D; for us 2D (images) is important

Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again

how can we be sure we are filling in the gaps correctly?



Undersampling

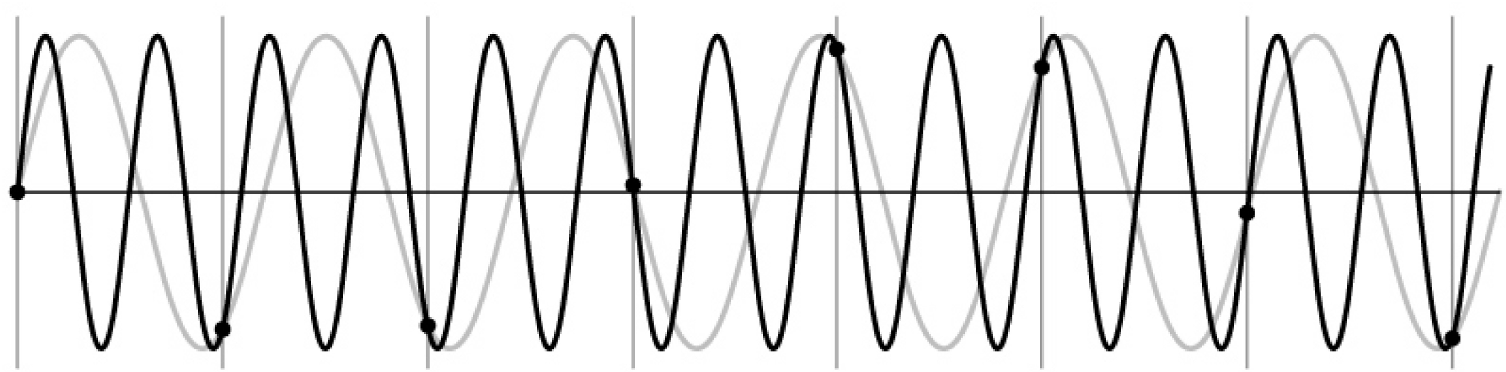
- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave

unsurprising result: information is lost

surprising result: indistinguishable from lower frequency

also was always indistinguishable from higher frequencies

aliasing: signals "traveling in disguise" as other frequencies

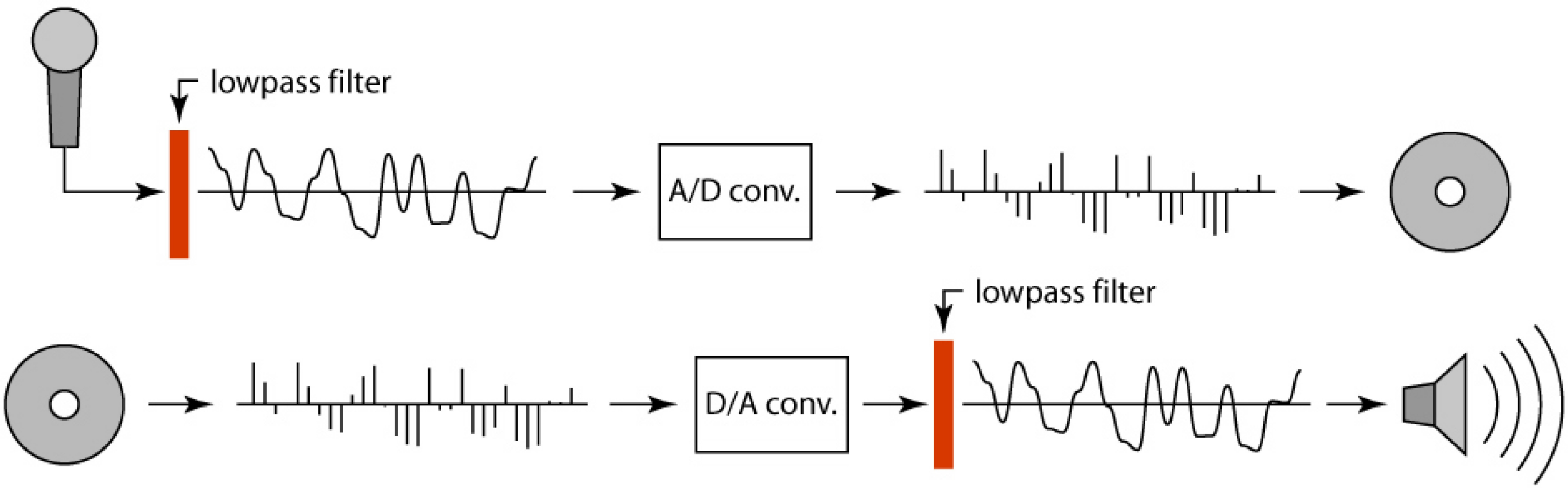


Preventing aliasing

- Introduce lowpass filters:

remove high frequencies leaving only safe, low frequencies

choose lowest frequency in reconstruction (disambiguate)



Linear filtering: a key idea

- Transformations on signals; e.g.:

bass/treble controls on stereo

blurring/sharpening operations in image editing

smoothing/noise reduction in tracking

- Key properties

linearity: filter(*f* + *g*) = filter(*f*) + filter(*g*)

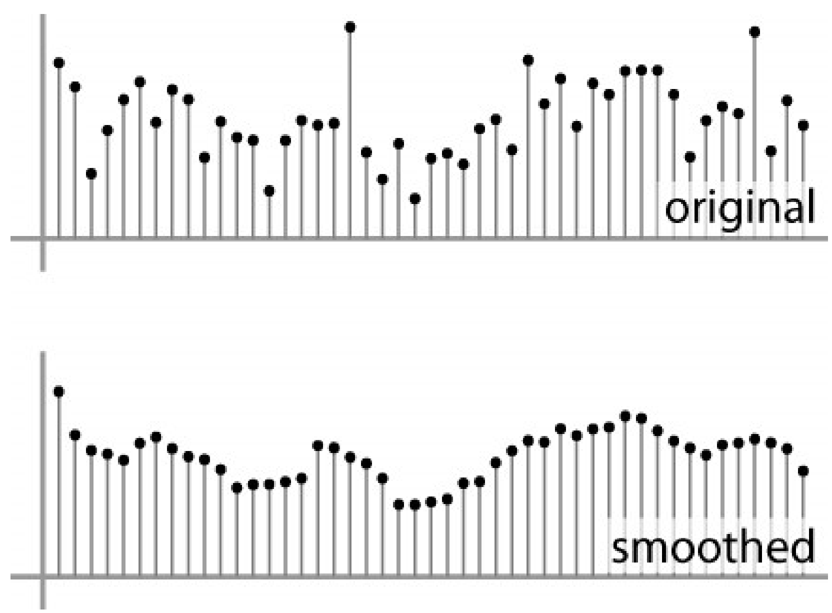
shift invariance: behavior invariant to shifting the input

  - delaying an audio signal
  - sliding an image around

- Can be modeled mathematically by *convolution*

Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



Convolution warm-up

- Same moving average operation, expressed mathematically:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

Discrete convolution

- Simple averaging:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

every sample gets the same weight

- Convolution: same idea but with *weighted* average

$$(a \star b)[i] = \sum_j a[j] b[i-j]$$

each sample gets its own weight (normally zero far away)

- This is all convolution is: it is a **moving weighted average**

Filters

- Sequence of weights *a*[*j*] is called a *filter*
- Filter is nonzero over its *region of support*

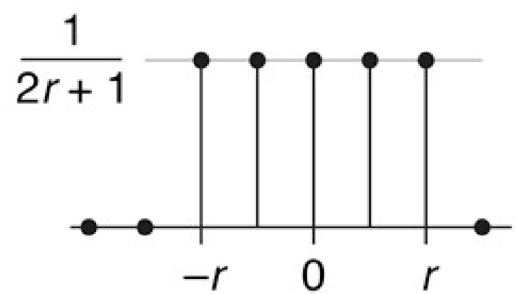
usually centered on zero: support radius *r*

- Filter is *normalized* so that it sums to 1.0

this makes for a weighted average, not just any old weighted sum

- Most filters are symmetric about 0

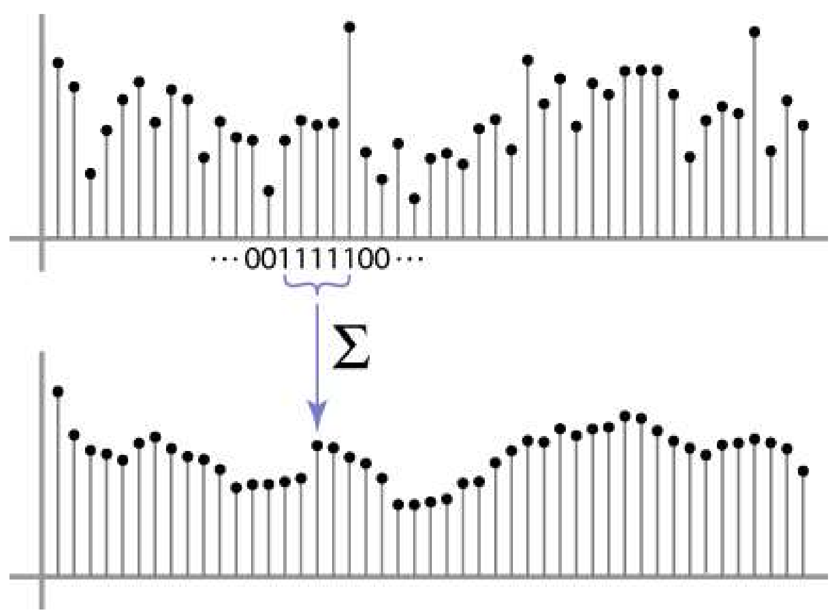
since for images we usually want to treat left and right the same



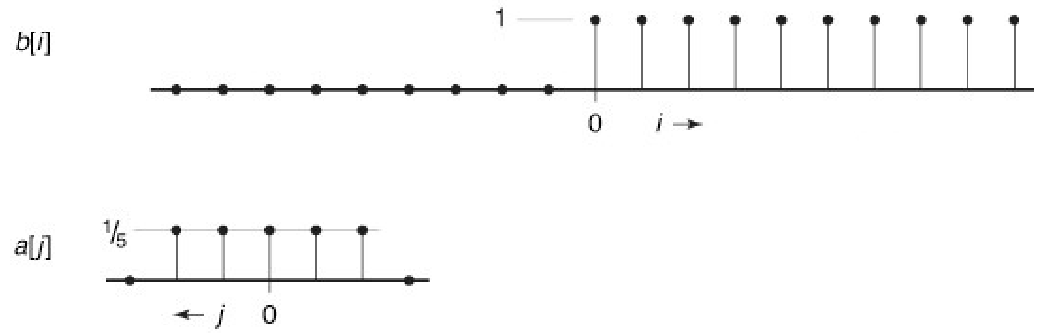
a box filter

Convolution and filtering

- Can express sliding average as convolution with a *box filter*
- $a_{\text{box}} = [\ldots, 0, 1, 1, 1, 1, 1, 0, \ldots]$

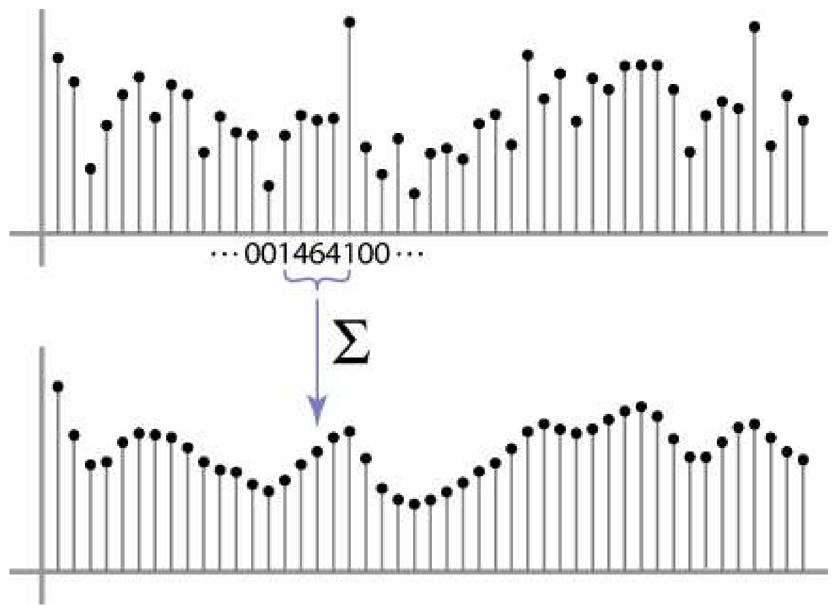


Example: box and step



Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) […, 1, 4, 6, 4, 1, …]/16



And in pseudocode…

function convolve(sequence a, sequence b, int r, int i)

$s = 0$

for $j = -r$ to r

$s = s + a[j]b[i - j]$

return s

Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation

commutative $a \star b = b \star a$

associative $a \star (b \star c) = (a \star b) \star c$

distributes over addition $a \star (b + c) = a \star b + a \star c$

scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$

identity: unit impulse *e* = [..., 0, 0, 1, 0, 0, ...]

$$a \star e = a$$

- Conceptually no distinction between filter and signal

Discrete filtering in 2D

- Same equation, one more index

$$(a \star b)[i,j] = \sum_{i',j'} a[i',j']b[i-i',j-j']$$

now the filter is a rectangle you slide around over a grid of numbers

- Commonly applied to images

blurring (using box, using gaussian, …)

sharpening (impulse minus blur)

- Usefulness of associativity

often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$

this is equivalent to applying one filter: $a * (b_1 * b_2 * b_3)$

And in pseudocode…

function convolve2d(filter2d a, filter2d b, int i, int j)
$s = 0$
$r = a$.radius
for $i' = -r$ to r **do**
    for $j' = -r$ to r **do**
        $s = s + a[i'][j']b[i - i'][j - j']$
return s


Philip Greenspun]
original ▲ |▼ box blur
sharpened ▲| ▼gaussian blur

Optimization: separable filters

- basic alg. is $O(r^2)$: large filters get expensive fast!
- definition: $a_2(x,y)$ is *separable* if it can be written as:

$$a_2[i,j] = a_1[i]a_1[j]$$

this is a useful property for filters because it allows factoring:

$$\begin{aligned}(a_2 \star b)[i,j] &= \sum_{i'}\sum_{j'} a_2[i',j']b[i-i',j-j'] \\ &= \sum_{i'}\sum_{j'} a_1[i']a_1[j']b[i-i',j-j'] \\ &= \sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i-i',j-j'] \right)\end{aligned}$$

Separable filtering

$$a_2[i,j] = a_1[i]a_1[j]$$

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

0	0	0	0	0
0	0	0	0	0
1	4	6	4	1
0	0	0	0	0
0	0	0	0	0

0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
0	0	4	0	0
0	0	1	0	0

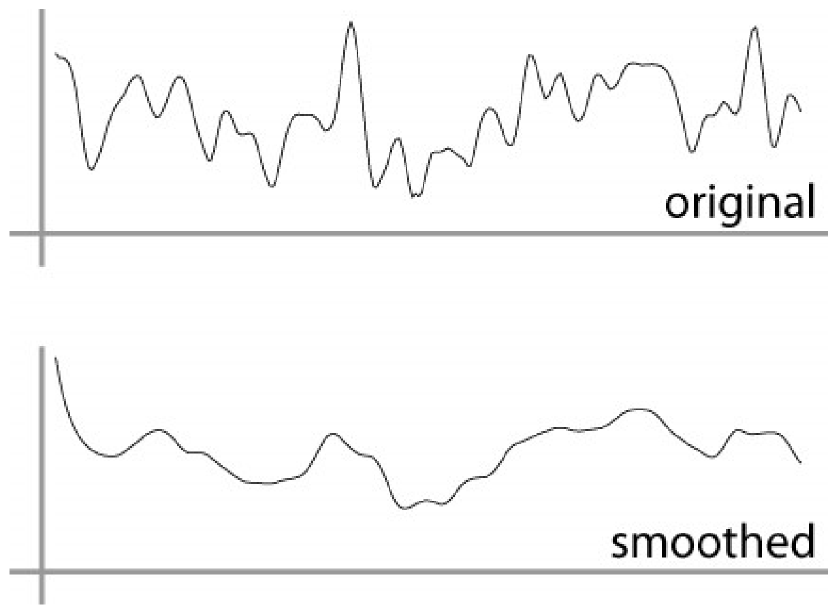
$$\overbrace{\sum_{i'} a_1[i']\left(\overbrace{\sum_{j'} a_1[j']b[i-i',j-j']}^{\text{first, convolve with this}}\right)}^{\text{second, convolve with this}}$$

Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well

output is continuous

integration replaces summation



Continuous convolution

- Sliding average expressed mathematically:

$$g_{\text{smooth}}(x) = \frac{1}{2r} \int_{x-r}^{x+r} g(t)dt$$

note difference in normalization (only for box)

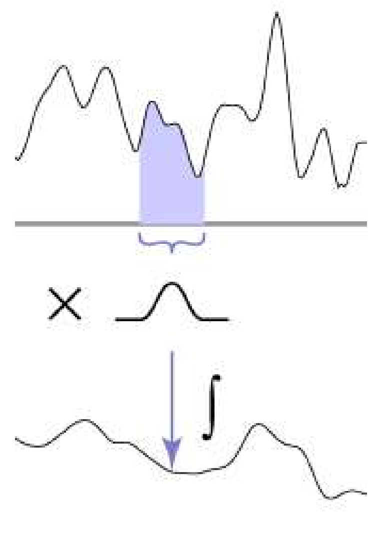
- Convolution just adds weights

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

weighting is now by a function

weighted integral is like weighted average

again bounds are set by support of *f(x)*



One more convolution

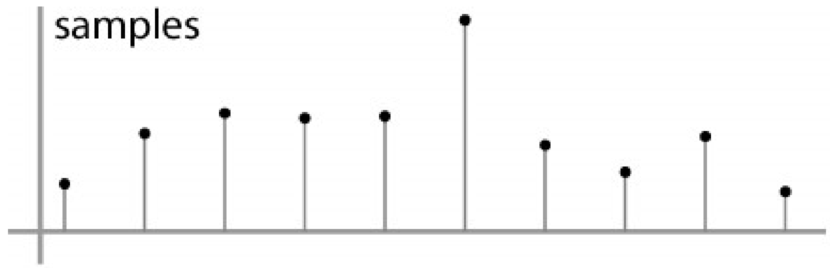
- Continuous–discrete convolution

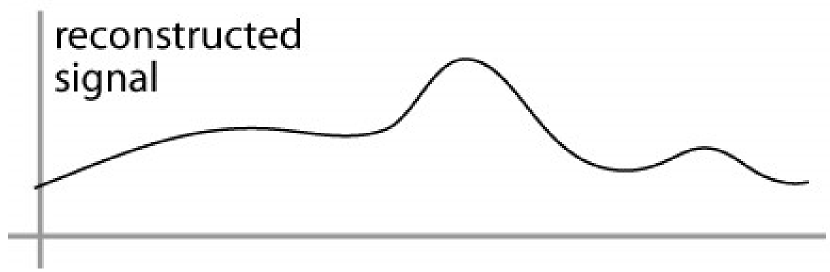
$$(a \star f)(x) = \sum_i a[i] f(x - i)$$

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

used for reconstruction and resampling

Continuous-discrete convolution





Resampling

- Changing the sample rate

in images, this is enlarging and reducing

- Creating more samples:

increasing the sample rate

“upsampling”

“enlarging”

- Ending up with fewer samples:

decreasing the sample rate

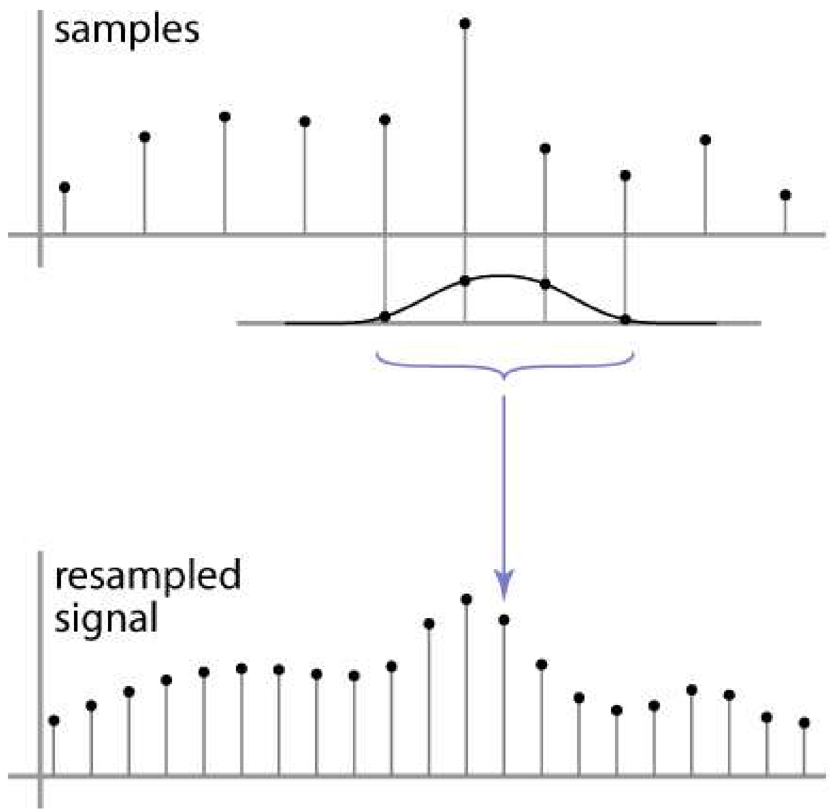
“downsampling”

“reducing”

Resampling

- Reconstruction creates a continuous function

forget its origins, go ahead and sample it



And in pseudocode…

function reconstruct(sequence a, filter f, real x)
$s = 0$
$r = f$.radius
for $i = \lceil x - r \rceil$ to $\lfloor x + r \rfloor$ **do**
    $s = s + a[i]f(x - i)$
return s

Cont.–disc. convolution in 2D

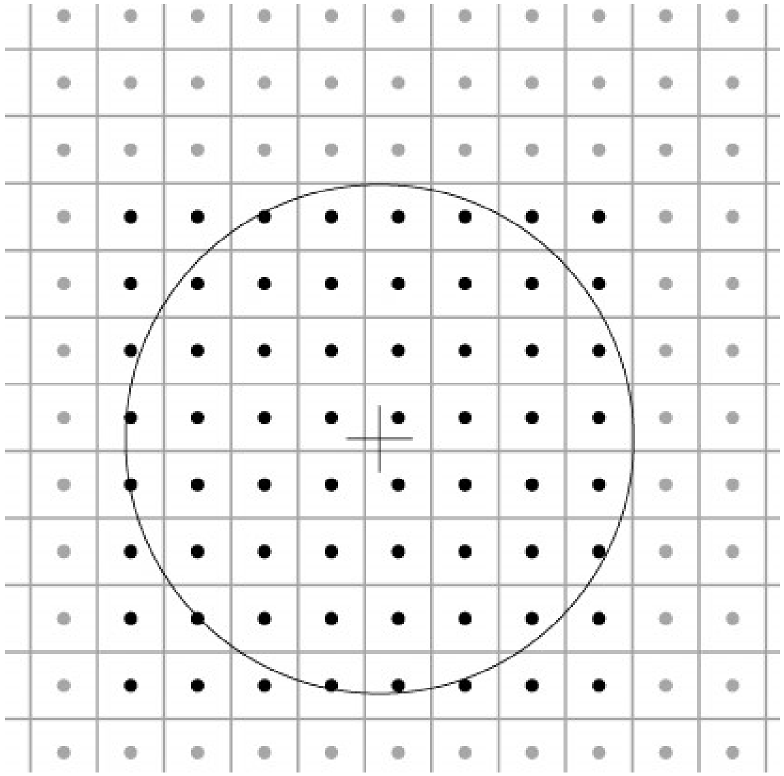
- same convolution—just two variables now

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

loop over nearby pixels,
average using filter weight

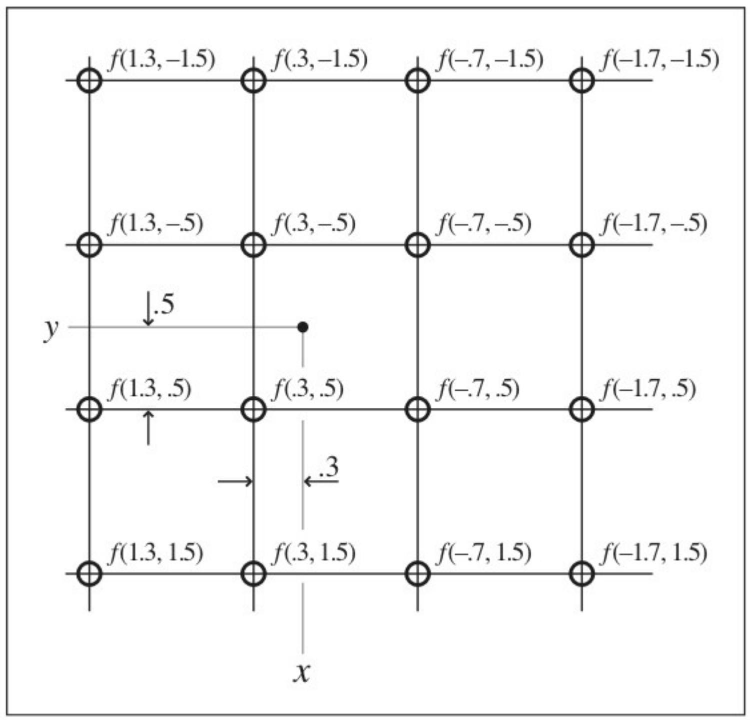
looks like discrete filter,
but offsets are not integers
and filter is continuous

remember placement of filter
relative to grid is variable



Cont.–disc. convolution in 2D

$$(a \star f)(x,y) = \sum_{i,j} a[i,j] f(x-i, y-j)$$



Separable filters for resampling

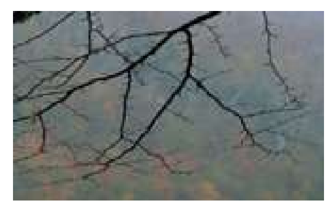
- just as in filtering, separable filters are useful

separability in this context is a statement about a continuous filter, rather than a discrete one:

$$f_2(x, y) = f_1(x) f_1(y)$$

- resample in two passes, one resampling each row and one resampling each column
- intermediate storage required: product of one dimension of src. and the other dimension of dest.
- same yucky details about boundary conditions

[Philip Greenspun]



two-stage resampling using a separable filter

A gallery of filters

- Box filter

Simple and cheap

- Tent filter

Linear interpolation

- Gaussian filter

Very smooth antialiasing filter

- B-spline cubic

Very smooth

- Catmull-rom cubic

Interpolating

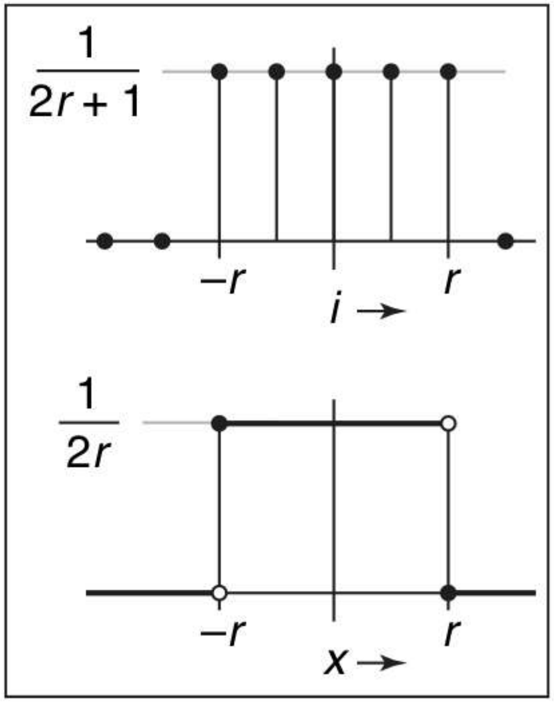
- Mitchell-Netravali cubic

Good for image upsampling

Box filter

$$a_{\text{box},r}[i] = \begin{cases} 1/(2r+1) & |i| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

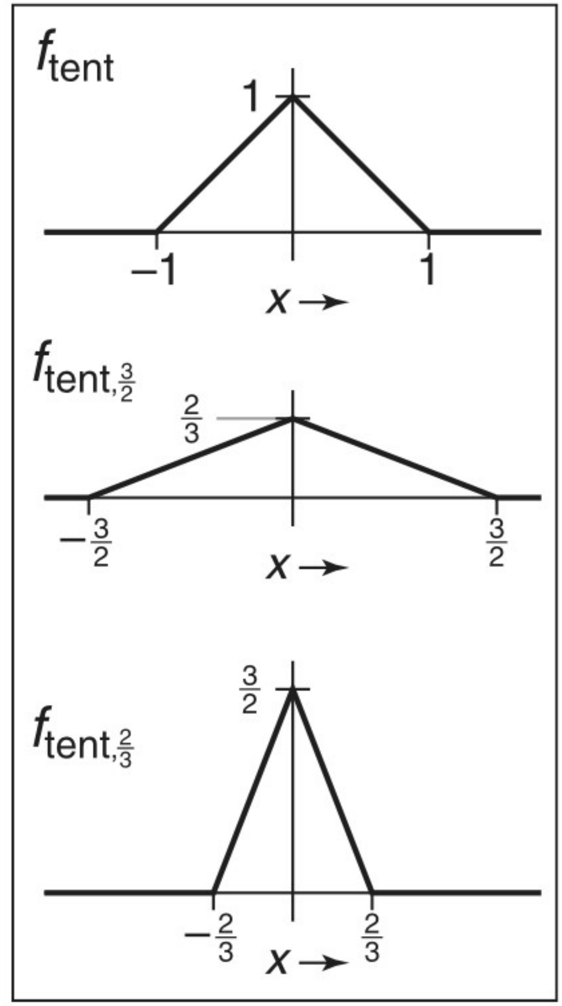
$$f_{\text{box},r}(x) = \begin{cases} 1/(2r) & -r \le x < r, \\ 0 & \text{otherwise.} \end{cases}$$



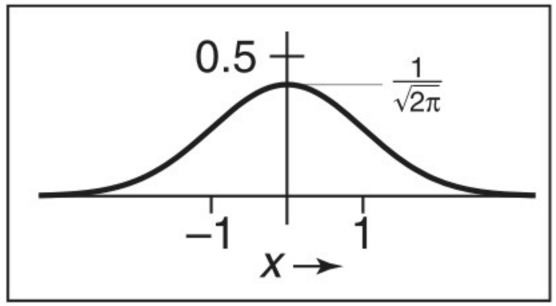
Tent filter

$$f_{\text{tent}}(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$f_{\text{tent},r}(x) = \frac{f_{\text{tent}}(x/r)}{r}.$$

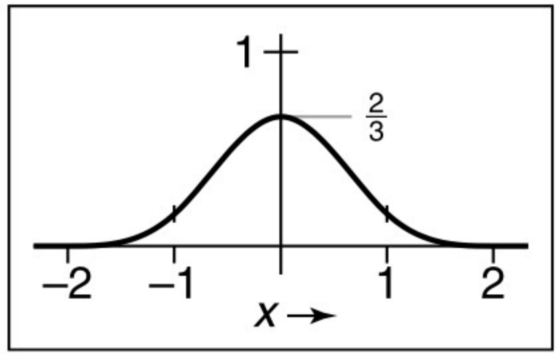


Gaussian filter



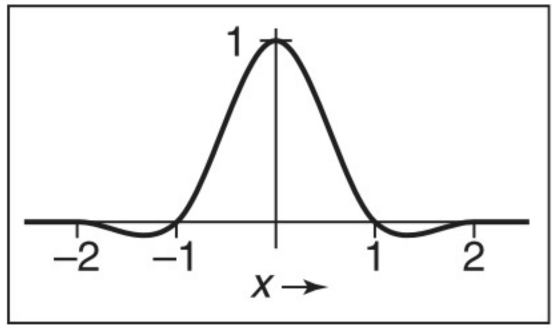
$$f_g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

B-Spline cubic



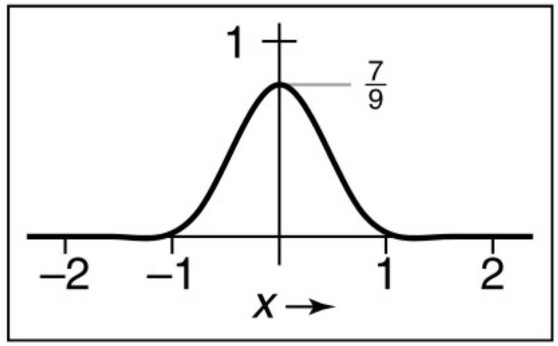
$$f_B(x) = \frac{1}{6}\begin{cases} -3(1-|x|)^3 + 3(1-|x|)^2 + 3(1-|x|) + 1 & -1 \leq x \leq 1, \\ (2-|x|)^3 & 1 \leq |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Catmull-Rom cubic



$$f_C(x) = \frac{1}{2} \begin{cases} -3(1-|x|)^3 + 4(1-|x|)^2 + (1-|x|) & -1 \le x \le 1, \\ (2-|x|)^3 - (2-|x|)^2 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Michell-Netravali cubic



$$f_M(x) = \frac{1}{3} f_B(x) + \frac{2}{3} f_C(x)$$

$$= \frac{1}{18} \begin{cases} -21(1-|x|)^3 + 27(1-|x|)^2 + 9(1-|x|) + 1 & -1 \le x \le 1, \\ 7(2-|x|)^3 - 6(2-|x|)^2 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Effects of reconstruction filters

- For some filters, the reconstruction process winds up implementing a simple algorithm
- Box filter (radius 0.5): nearest neighbor sampling

box always catches exactly one input point

it is the input point nearest the output point

so output[i, j] = input[round(x(i)), round(y(j))]
  x(i) computes the position of the output coordinate i on the input grid

- Tent filter (radius 1): linear interpolation

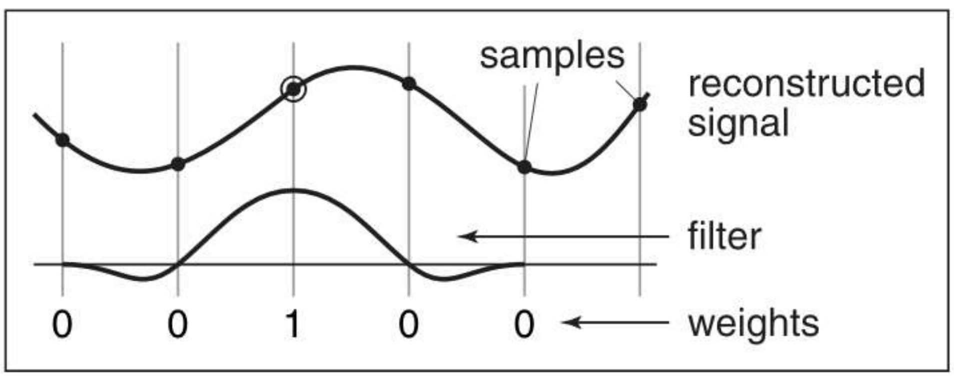
tent catches exactly 2 input points

weights are a and (1 – a)

result is straight-line interpolation from one point to the next

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

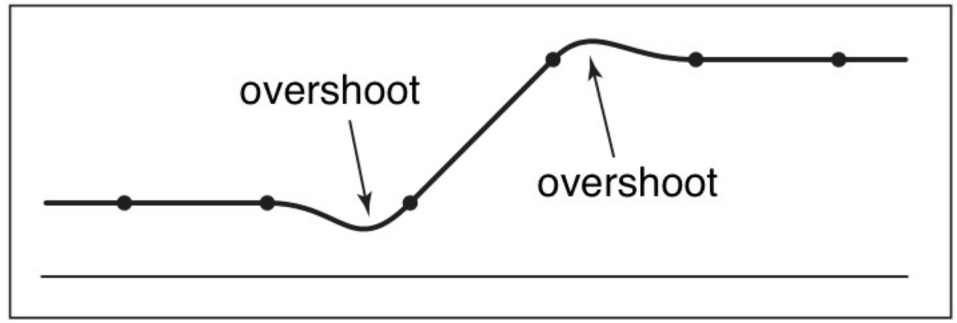


interpolating filter used for reconstruction

Ringing, overshoot, ripples

- Overshoot

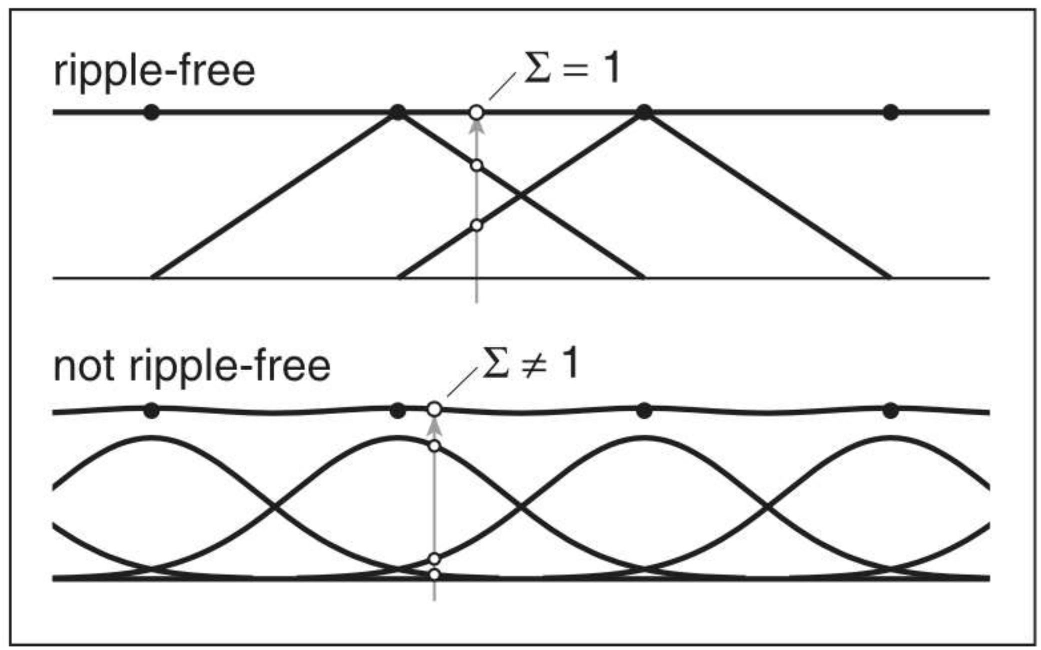
caused by
negative filter
values



- Ripples

constant in,
non-const. out

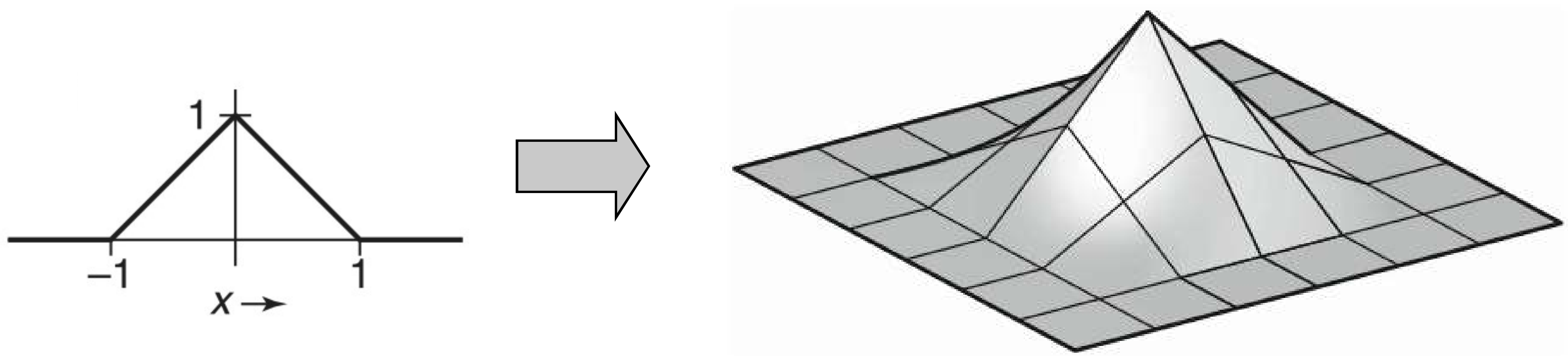
ripple free when:

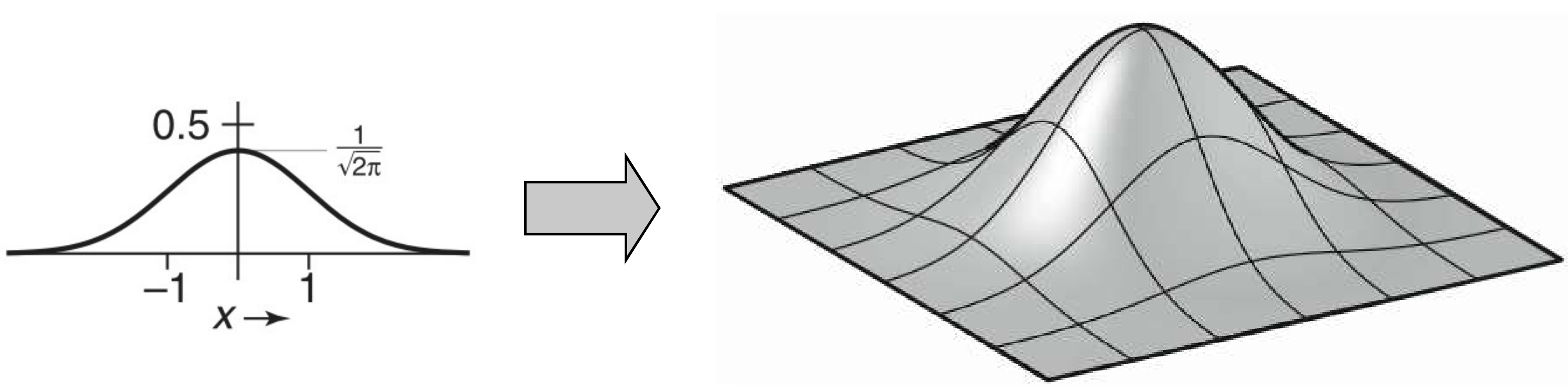


$$\sum_i f(x+i) = 1 \quad \text{for all } x.$$

Constructing 2D filters

- Separable filters (most common approach)





Yucky details

- What about near the edge?

the filter window falls off the edge of the image

need to extrapolate

methods:

  - clip filter (black)
  - wrap around
  - copy edge
  - reflect across edge
  - vary filter near edge

[Philip Greenspun]

Reducing and enlarging

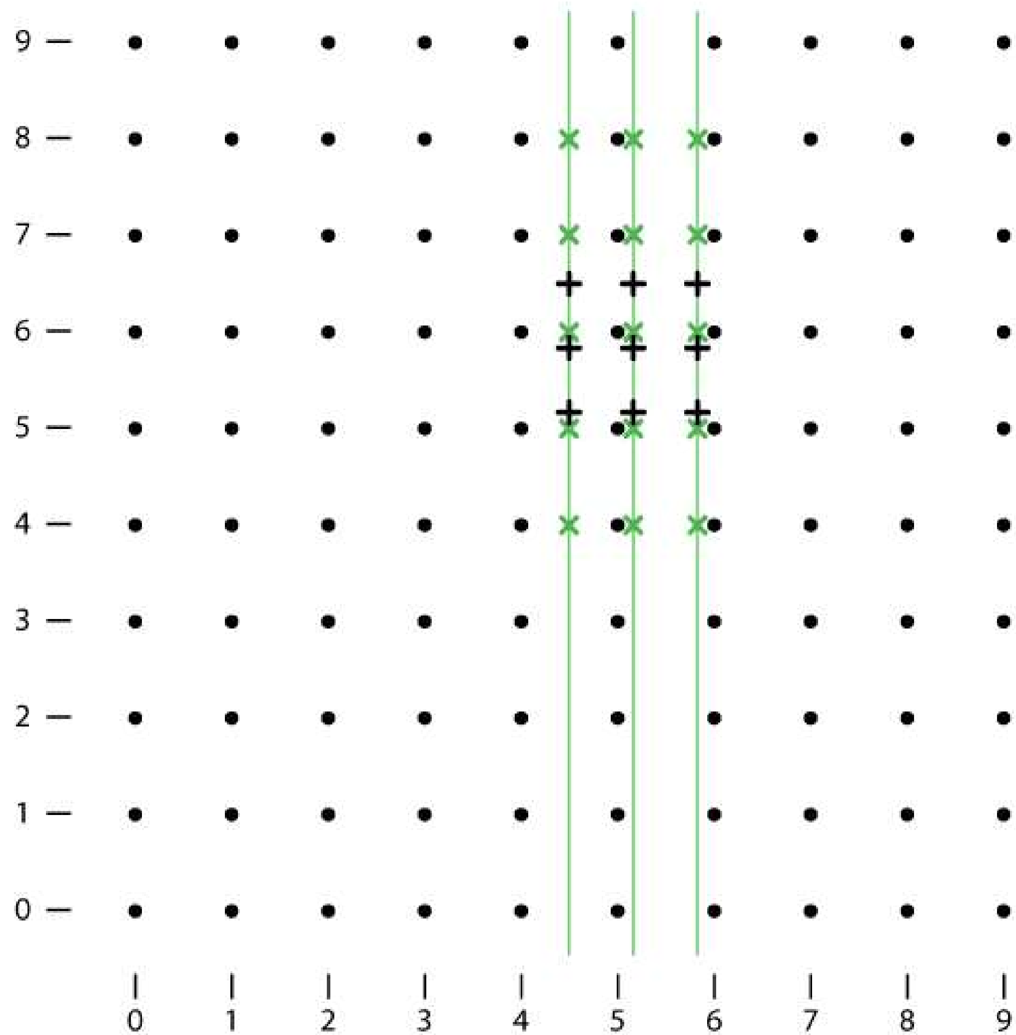
- Very common operation

devices have differing resolutions

applications have different memory/quality tradeoffs

- Also very commonly done poorly
- Simple approach: drop/replicate pixels
- Correct approach: use resampling

Resampling example



Reducing and enlarging

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- Simple approach: drop/replicate pixels
- Correct approach: use resampling

1000 pixel width

[Philip Greenspun]

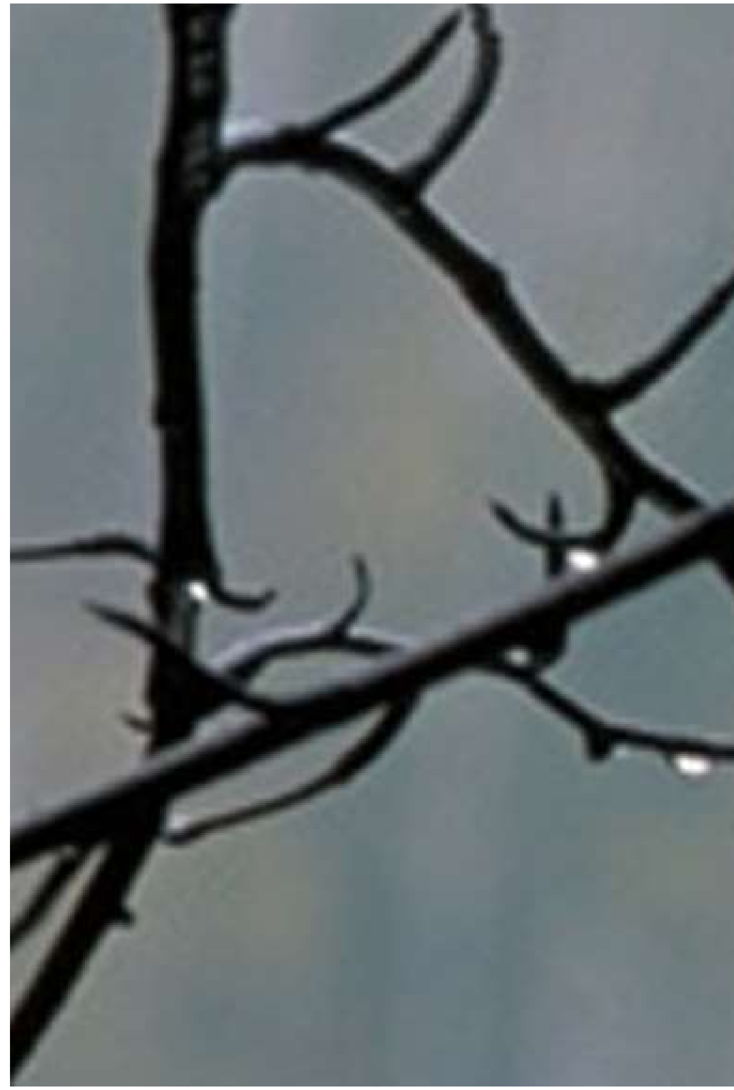
[Philip Greenspun]

by dropping pixels

gaussian filter

250 pixel width

box reconstruction filter



bicubic reconstruction filter

[Philip Greenspun]

4000 pixel width

Types of artifacts

- Garden variety

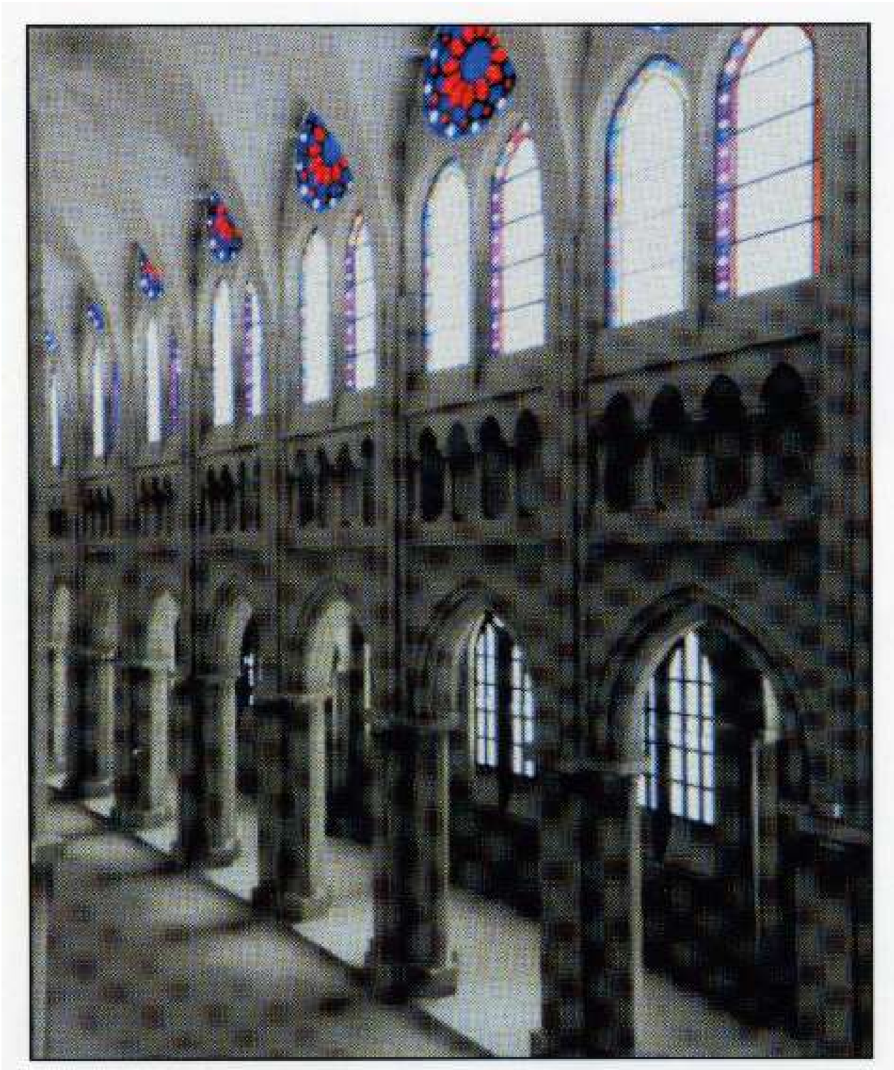
what we saw in this natural image

fine features become jagged or sparkle

- Moiré patterns

[Hearn & Baker cover]

600ppi scan of a color halftone image



by dropping pixels

gaussian filter

[Hearn & Baker cover]

downsampling a high resolution scan

Types of artifacts

- Garden variety

what we saw in this natural image

fine features become jagged or sparkle

- Moiré patterns

caused by repetitive patterns in input

produce large-scale artifacts; highly visible

- These artifacts are *aliasing* just like in the audio example earlier
- How do I know what filter is best at preventing aliasing?

practical answer: experience

theoretical answer: there is another layer of cool math behind all this

  - based on Fourier transforms
  - provides much insight into aliasing, filtering, sampling, and reconstruction